# Math 290 ELEMENTARY LINEAR ALGEBRA REVIEW OF LECTURES - IX 

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§9. Gaussian Elimination.
Last time I threw the system

$$
\left\{\begin{aligned}
x+y+z & =2 \\
-x+3 y+2 z & =8 \\
4 x+y & =4
\end{aligned}\right.
$$

(= Example 5 from "Review of Lectures — VIII"). We solved it brute-force:

$$
(x, y, z)=\left(\begin{array}{lll}
0, & 4, & -2
\end{array}\right)
$$

The method employed therein is called "Gaussian elimination". Today's discussion features how to frame it in the context of matrix operations. Let's set up the problem:

Problem 1. Solve the following system (the same as above)

$$
\left\{\begin{aligned}
x+y+z & =2 \\
-x+3 y+2 z & =8 \\
4 x+y & =4
\end{aligned}\right.
$$

$\underline{\underline{u s i n g ~ m a t r i c e s . ~}}$
Solution using matrices. Construct the so-called augmented matrix
(*)

$$
\left[\begin{array}{ccc|c}
1 & 1 & 1 & 2 \\
-1 & 3 & 2 & 8 \\
4 & 1 & 0 & 4
\end{array}\right]
$$

As you can see, I created this matrix basically by way of just cutting-and-pasting the numbers residing in the equations at the respective spots of the array. Vertical rule is there to mutually separate the data coming from the two sides of the system. Now, if you ignore the vertical rule, this is a $3 \times 4$ matrix - three rows, four columns: Who says we should preclude matrices that have different numbers of rows and columns? No, nobody. So let's adopt them (why not?) Also, the vertical rule actually doesn't play any role. So let's just get rid of it.
Goal. Re-enact the steps in page 15-16 of "Review of Lectures - VIII", using matrices, and ultimately reduce $(*)$ to (\#) below (as to why (\#), see page 5):

$$
\begin{align*}
& {\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & -2
\end{array}\right] .} \\
& {\left[\begin{array}{|cccc|}
\hline 1 & 1 & 1 & 2 \\
\hline \hline-1 & 3 & 2 & 8 \\
4 & 1 & 0 & 4 \\
\hline
\end{array}\right] \cdot(-2) \quad \square-\text { add up }} \\
& \rightarrow\left[\begin{array}{cccc}
1 & 1 & 1 & 2 \\
\hline-3 & 1 & 0 & 4 \\
4 & 1 & 0 & 4
\end{array}\right] \longleftarrow
\end{align*}
$$

Step 1.
(Keep the original top row intact.)

Step 2.

$$
\begin{aligned}
& {\left[\begin{array}{cccc}
1 & 1 & 1 & 2 \\
\hline-3 & 1 & 0 & 4 \\
\hline 4 & 1 & 0 & 4 \\
\hline
\end{array}\right] \cdot(-1) \quad \square-\text { add up } } \\
\rightarrow & {\left[\begin{array}{cccc}
1 & 1 & 1 & 2 \\
-3 & 1 & 0 & 4 \\
7 & 0 & 0 & 0 \\
\hline
\end{array}\right] }
\end{aligned}
$$

(Keep the original middle row intact.)

Step 3.

$$
\begin{aligned}
& {\left[\begin{array}{cccc}
1 & 1 & 1 & 2 \\
-3 & 1 & 0 & 4 \\
7 & 0 & 0 & 0 \\
\hline
\end{array}\right] } \\
& \rightarrow {\left[\begin{array}{cccc}
1 & 1 & 1 & 2 \\
-3 & 1 & 0 & 4 \\
1 & 0 & 0 & 0 \\
\hline
\end{array}\right] }
\end{aligned}
$$

Step 4.


$$
\rightarrow\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\hline 0 & 1 & 0 & 4 \\
\hline 1 & 1 & 1 & 2
\end{array}\right]
$$

Step 5.
(Keep the original top row intact.)

## Step 6.

$$
\begin{aligned}
& {\left[\begin{array}{|cccc|}
\hline 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 4 \\
\hline 1 & 1 & 1 & 2 \\
\hline
\end{array}\right] \cdot(-1) \longrightarrow \text { add up }} \\
& \rightarrow\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 4 \\
\hline 0 & 1 & 1 & 2
\end{array}\right] \longleftarrow
\end{aligned}
$$

(Keep the original top row intact.)

Step 7.

$$
\begin{aligned}
& {\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
\hline 0 & 1 & 0 & 4 \\
\hline \hline 0 & 1 & 1 & 2
\end{array}\right] \cdot(-1) \longrightarrow-\text { add up }} \\
& {\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 4 \\
\hline 0 & 0 & 1 & -2
\end{array}\right] \longleftarrow}
\end{aligned}
$$

(Keep the original middle row intact.)

Now, let's take a look at this last matrix. This is exactly what we were shooting for:
(\#)
\(\frac{\left[$$
\begin{array}{ccc|c}{\left[\begin{array}{ccc}1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1\end{array}
$$\right.} \& 0 <br>

4\end{array}\right]}{\|}\)| $\\|$ |
| :--- |
| $I$ |.

The boxed part is the identity matrix $I$. Agree with the following:
"Just like
(*)

$$
\left[\begin{array}{cccc}
1 & 1 & 1 & 2 \\
-1 & 3 & 2 & 8 \\
4 & 1 & 0 & 4
\end{array}\right]
$$

is what the system

$$
\left\{\begin{aligned}
x+y+z & =2, \\
-x+3 y+2 z & =8 \\
4 x+y & =4
\end{aligned}\right.
$$

(= the system in the original problem) is confined as,
(\#)
$\frac{\left[\begin{array}{lll|}{\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right.} & 0 \\ \| & -2\end{array}\right]}{I}$
is what the new system
is confined as".

Since the matrix $(*)$ was reduced to (\#) through the row operations (Step 1-Step 7), $x=0 ; \quad y=4$, and $z=-2$, are the final answers.

## Answer for Problem 1:

$$
(x, y, z)=\left(\begin{array}{lll}
0, & 4, & -2
\end{array}\right) .
$$

- Row operations. How to invert a $(3 \times 3)$ matrix.

The above solution prompts you to raise the following set of questions:

Question 1a. What sort of operations are allowed at every step of the reduction process $(*) \rightarrow(\#)$ ?

Question 1b. Is there a name for something like (\#)?

Question 1c. Is it always feasible to reduce something like (*) to something like (\#)?

Question 2a. Basically, the system

$$
\left\{\begin{aligned}
x+y+z & =2 \\
-x+3 y+2 z & =8 \\
4 x+y & =4
\end{aligned}\right.
$$

is like


This is of the form $A \boldsymbol{x}=\boldsymbol{b}$. We already know how to invert $A$.
Then, just like the $2 \times 2$ case, the "golden rule"

$$
A x=b \quad \underset{\substack{\text { can solve, } \\ \text { if } \operatorname{det} A \neq 0}}{\Longrightarrow} \quad x=A^{-1} b
$$

should take care of the problem. Is that correct? Then isn't it true that the above 'solution' is redundant? Why do we have to learn that?

Question 2b. That said, computation of $A^{-1}$ is very cumbersome. So, basically can I understand that the above 'solution' replaces the "golden rule" cited above? Then what is the use of the inverse of a matrix? Now our knowledge how to invert $A$ seems redundant.

Question 3. Back to Question 2a, what happens if $A^{-1}$ does not exist? What can one say about the root of the system?

- Let's sort all these out.


## Answer to Question 1a.

Three operations, called elementary row operations, are allowed:

- multiply a scalar $t$ to one entire row, where $t \neq 0$.
$\circ \underline{\underline{\text { add }}}[\underline{\underline{t(\text { a scalar }) \text { times row } \# \mathrm{a}}}] \underline{\underline{\text { to row } \# \mathrm{~b}}}$ (while keeping row \#a intact).
- Interchange row \#a and row \#b.


## Answer to Question 1b.

Yes: Reduced row echelon form. More precisely, any $3 \times 4$ matrix whose shape falls into one of the following fifteen types is called a reduced row echelon form, where each of the ' $*$ ' spots is filled by an arbitrary number:

$$
\left.\begin{array}{lll}
{\left[\begin{array}{llll}
1 & 0 & 0 & * \\
0 & 1 & 0 & * \\
0 & 0 & 1 & *
\end{array}\right],} & {\left[\begin{array}{llll}
1 & 0 & * & 0 \\
0 & 1 & * & 0 \\
0 & 0 & 0 & 1
\end{array}\right],} & {\left[\begin{array}{llll}
1 & * & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right],}
\end{array} \begin{array}{lll}
1 & 0 & *
\end{array} *\right)\left[\begin{array}{llll}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right],
$$

- Any reduced echelon form, in any size (not just $3 \times 4$ ), obeys the following:
(i) If you read off each row from left to right, then either it starts with 1 , or it starts with 0 and 0 repeats until 1 shows up at some point, or the whole row is entirely 0 . The first 1 from the left, if any, is called the leading 1.
(ii) A column that contains a leading 1 has 0 everywhere else.
(iii) By (ii), if there are two leading 1 s, they cannot coexist in the same column. In that case the lower one sits further right to the upper one.
(iv) A row that consists entirely 0 , if any, are grouped together at the bottom.

The properties (i), (ii), (iii) and (iv) characterize reduced echelon forms in any size.

Example 1. The following $3 \times 4$ matrices are in reduced row echelon form:

$$
\left[\begin{array}{cccc}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & -3
\end{array}\right], \quad\left[\begin{array}{llll}
0 & 1 & 4 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array}\right], \quad\left[\begin{array}{llll}
1 & 2 & 3 & 6 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right], \quad\left[\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] .
$$

Exercise 1. Which one of the following are in reduced row echelon form?
(1) $\left[\begin{array}{llll}0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1\end{array}\right]$,
(2) $\left[\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$,
(3) $\left[\begin{array}{lll}0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]$,
(4) $\left[\begin{array}{llll}0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0\end{array}\right]$,
(5) $\left[\begin{array}{llll}0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1\end{array}\right]$,
(6) $\left[\begin{array}{ll}1 & 0 \\ 0 & 1 \\ 0 & 0\end{array}\right]$,

[^0](10) $\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]$,

$$
\left[\begin{array}{lllll}
0 & 0 & 1 & 0 & 0  \tag{12}\\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2 & 0
\end{array}\right] .
$$

Exercise 2. List up all possible $3 \times 6$ reduced row echelon forms. Use ' $*$ ' to denote an arbitrary entry. There are forty two (42) different types.

- Twenty (20) of those have three leading 1 s (the rank is 3 ).
- Fifteen (15) of those have two leading 1s (the rank is 2 ).
- $\operatorname{Six}$ (6) of those have one leading 1 (the rank is 1 ).
- One (1) of those has no leading 1 s (the matrix consists entirely of 0 s .)


## Answer to Question 1c.

No, it is not always true that any augmented $3 \times 4$ matrix can be reduced to


However, it is true that any augmented $3 \times 4$ matrix can be reduced to a reduced row echelon form (either one of the fifteen in the list in page 7). More generally, it is true that any matrix can be reduced to a reduced row echelon form of the same size via elementary row operations.

Answer to Questions 2a \& 3. It is absolutely correct, that the "golden rule"

$$
A x=b \quad \underset{\substack{\text { can solve, } \\ \text { if det } A \neq 0}}{\Longrightarrow} \quad x=A^{-1} b
$$

applies to Problem 1. The 'solution' in pages $1-5$ above gives an alternative way to pull the same answer. That said, there are pros and cons: The 'golden rule' works only when $\operatorname{det} A \neq 0$. In the above, we didn't know whether $\operatorname{det} A \neq 0$ beforehand. The method in pages $1-5$ is at our disposal even when we don't know whether $\operatorname{det} A \neq 0$ beforehand. We are going to address equations $A \boldsymbol{x}=\boldsymbol{b}$ with $\operatorname{det} A=0$, and also equations $A \boldsymbol{x}=\boldsymbol{b}$ where $A$ is not in square size (different number of rows and columns) in the forthcoming lectures.

## Answer to Question 2b.

Now, what I said above in 'Answer to Question 2a' might appear to suggest that the notion of the inverse of matrices is redundant. The truth is, 'that's not accurate'. The notion of the inverse is absolutely indispensable. The exact same method - to reduce a given matrix to a reduced row echelon form - can be applied to calculate the inverse $A^{-1}$ of a given matrix $A$. It is summarized in Formula 1 below. Our solution in page $1-5$ did not make a direct reference to $A^{-1}$. Yet as you can clearly see below, the gist of what we have worked out above essentially amounts to calculating $A^{-1}$, and thereby the root $A^{-1} \boldsymbol{b}$.

Formula 1. For a $3 \times 3$ matrix $A=\left[\begin{array}{lll}a_{1} & a_{2} & a_{3} \\ b_{1} & b_{2} & b_{3} \\ c_{1} & c_{2} & c_{3}\end{array}\right]$, construct

$$
\left[\begin{array}{l|l}
A & I
\end{array}\right]=\left[\begin{array}{llllll}
a_{1} & a_{2} & a_{3} & 1 & 0 & 0 \\
b_{1} & b_{2} & b_{3} & 0 & 1 & 0 \\
c_{1} & c_{2} & c_{3} & 0 & 0 & 1
\end{array}\right]
$$

This is a $3 \times 6$ matrix.
(1) If the reduced row echelon form of $[A \mid I]$ is of the form

$$
[I \mid B]=\left[\begin{array}{cccccc}
1 & 0 & 0 & p_{1} & p_{2} & p_{3} \\
0 & 1 & 0 & q_{1} & q_{2} & q_{3} \\
0 & 0 & 1 & r_{1} & r_{2} & r_{3}
\end{array}\right]
$$

then $\operatorname{det} A \neq 0$, and moreover

$$
B=\left[\begin{array}{lll}
p_{1} & p_{2} & p_{3} \\
q_{1} & q_{2} & q_{3} \\
r_{1} & r_{2} & r_{3}
\end{array}\right]
$$

is the inverse of $A: \quad B=A^{-1}$.
(2) If the reduced row echelon form of $[A \mid I]$ is of the form

$$
\left[\begin{array}{llllll}
* & * & * & * & * & * \\
* & * & * & * & * & * \\
0 & 0 & 0 & * & * & *
\end{array}\right],
$$

then $\operatorname{det} A=0$. In this case, $A^{-1}$ does not exist.

Example 2. Let us find $A^{-1}$ for

$$
A=\left[\begin{array}{ccc}
-2 & 2 & 3 \\
1 & -1 & 0 \\
0 & 1 & 4
\end{array}\right]
$$

if exists.
Step 1. Form

$$
\left[\begin{array}{l|l}
A & I
\end{array}\right]=\left[\begin{array}{cccccc}
-2 & 2 & 3 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 \\
0 & 1 & 4 & 0 & 0 & 1
\end{array}\right]
$$

Step 2. Apply Gaussian elimination method (the three operations in page 7, "Answer to Question 1a") to reduce this matrix to a reduced row echelon form (in what follows, top row; middle row, and bottom row, will be referred to as (row 1); (row 2), and (row 3), respectively).

$$
\begin{aligned}
& {\left[\begin{array}{cccccc}
-2 & 2 & 3 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 \\
0 & 1 & 4 & 0 & 0 & 1
\end{array}\right] \rightarrow\left[\begin{array}{cccccc}
1 & -1 & -3 / 2 & -1 / 2 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 \\
0 & 1 & 4 & 0 & 0 & 1
\end{array}\right]} \\
& [(-1 / 2) \text { was multiplied to (row } 1)] \\
& \rightarrow\left[\begin{array}{cccccc}
1 & -1 & -3 / 2 & -1 / 2 & 0 & 0 \\
0 & 0 & 3 / 2 & 1 / 2 & 1 & 0 \\
0 & 1 & 4 & 0 & 0 & 1
\end{array}\right] \\
& {[(-1) \text { times (row 1) was added to (row 2) }]} \\
& \rightarrow\left[\begin{array}{cccccc}
1 & -1 & -3 / 2 & -1 / 2 & 0 & 0 \\
0 & 1 & 4 & 0 & 0 & 1 \\
0 & 0 & 3 / 2 & 1 / 2 & 1 & 0
\end{array}\right] \\
& {[\text { (row 2) and (row 3) were interchanged }]}
\end{aligned}
$$

$$
\rightarrow\left[\begin{array}{cccccc}
1 & 0 & 5 / 2 & -1 / 2 & 0 & 1 \\
0 & 1 & 4 & 0 & 0 & 1 \\
0 & 0 & 3 / 2 & 1 / 2 & 1 & 0
\end{array}\right]
$$

$[$ (row 2) was added to (row 1) $]$

$$
\rightarrow\left[\begin{array}{cccccc}
1 & 0 & 5 / 2 & -1 / 2 & 0 & 1 \\
0 & 1 & 4 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 / 3 & 2 / 3 & 0
\end{array}\right]
$$

$[(2 / 3)$ was multiplied to (row 3$)]$

$$
\rightarrow\left[\begin{array}{cccccc}
1 & 0 & 0 & -4 / 3 & -5 / 3 & 1 \\
0 & 1 & 0 & -4 / 3 & -8 / 3 & 1 \\
0 & 0 & 1 & 1 / 3 & 2 / 3 & 0
\end{array}\right]
$$

$[(-5 / 2)$ times (row 3$)$ was added to (row 1 );
$(-4)$ times (row 3 ) was added to (row 2)].
So, by part (1) of Formula 1, $A^{-1}$ indeed exists, and it is

$$
A^{-1}=\left[\begin{array}{ccc}
-4 / 3 & -5 / 3 & 1 \\
-4 / 3 & -8 / 3 & 1 \\
1 / 3 & 2 / 3 & 0
\end{array}\right] .
$$

Exercise 3. Verify that the result for $A^{-1}$ in Example 2 above is correct, by way of physically calculating $A^{-1} A$ (or alternatively, $A A^{-1}$ ). If the outcome equals $I$, then this answer is indeed correct. (The proof of the fact that this is indeed the correct checking method is pending.)

Example 3. Let us find $B^{-1}$ for

$$
B=\left[\begin{array}{ccc}
1 & 2 & -1 \\
3 & 7 & -10 \\
7 & 16 & -21
\end{array}\right]
$$

if exists.

Step 1. Form

$$
\left[\begin{array}{l|l}
B & I
\end{array}\right]=\left[\begin{array}{cccccc}
1 & 2 & -1 & 1 & 0 & 0 \\
3 & 7 & -10 & 0 & 1 & 0 \\
7 & 16 & -21 & 0 & 0 & 1
\end{array}\right]
$$

Step 2. Apply Gaussian elimination method (the three operations in page 7, "Answer to Question 1a") to reduce this matrix to a reduced row echelon form:

$$
\begin{aligned}
& {\left[\begin{array}{cccccc}
1 & 2 & -1 & 1 & 0 & 0 \\
3 & 7 & -10 & 0 & 1 & 0 \\
7 & 16 & -21 & 0 & 0 & 1
\end{array}\right] \rightarrow\left[\begin{array}{cccccc}
1 & 2 & -1 & 1 & 0 & 0 \\
0 & 1 & -7 & -3 & 1 & 0 \\
0 & 2 & -14 & -7 & 0 & 1
\end{array}\right]} \\
& {[(-3) \text { times (row 1) was added to (row 2); }} \\
& (-7) \text { times (row 1) was added to (row 3) }] \\
& \rightarrow\left[\begin{array}{cccccc}
1 & 2 & -1 & 1 & 0 & 0 \\
0 & 1 & -7 & -3 & 1 & 0 \\
0 & 0 & 0 & -1 & -2 & 1
\end{array}\right] . \\
& {[(-2) \text { times (row 2) was added to (row 3)]. }}
\end{aligned}
$$

This last matrix is of the form

$$
\left[\begin{array}{llllll}
* & * & * & * & * & * \\
* & * & * & * & * & * \\
0 & 0 & 0 & * & * & *
\end{array}\right] .
$$

So, by part (2) of Formula 1, $B^{-1}$ does not exist.

Exercise 4. For $B$ in Example 3 above, verify that $B^{-1}$ does not exist, independently of the above, by way of calculating $\operatorname{det} B$. If the outcome equals 0 , then the above conclusion, that $B^{-1}$ does not exist, is indeed correct.

Exercise 5. Use Formula 1 to invert each of the six matrices (1-6) in Exercise 5, page 14 of "Review of Lectures - III". Verify that you get the same answer for each of $(1-6)$. Let me duplicate them below:
(1) $A=\left[\begin{array}{ccc}2 & 1 & -2 \\ 5 & -4 & -1 \\ 1 & -3 & 4\end{array}\right]$.
(2) $A=\left[\begin{array}{ccc}1 & 3 & 1 \\ 2 & 4 & 1 \\ 1 & -2 & -2\end{array}\right]$.
(3) $A=\left[\begin{array}{ccc}3 & 4 & -4 \\ 2 & 1 & 4 \\ -2 & 4 & 1\end{array}\right]$.
(4) $A=\left[\begin{array}{ccc}3 & 5 & 10 \\ 3 & 1 & 6 \\ -2 & -2 & -6\end{array}\right]$.
(5) $A=\left[\begin{array}{ccc}1 & \sqrt{2} & \sqrt{3} \\ \sqrt{2} & \frac{-2-3 \sqrt{6}}{5} & \frac{6-\sqrt{6}}{5} \\ \sqrt{3} & \frac{6-\sqrt{6}}{5} & \frac{-3-2 \sqrt{6}}{5}\end{array}\right]$.
(6) $A=\left[\begin{array}{ccc}\frac{2+3 \sqrt{2}}{8} & \frac{-2 \sqrt{3}+\sqrt{6}}{8} & \frac{\sqrt{6}}{4} \\ \frac{-2 \sqrt{3}+\sqrt{6}}{8} & \frac{6+\sqrt{2}}{8} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2}\end{array}\right]$.


[^0]:    $\left[\begin{array}{ll}0 & 1 \\ 0 & 1\end{array}\right]$,
    (8) $\left[\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right]$,
    (9) $\left[\begin{array}{ccc}1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1\end{array}\right]$,

    $$
    \left[\begin{array}{lll}
    1 & 1 & 1 \\
    1 & 1 & 1 \\
    1 & 1 & 1
    \end{array}\right]
    $$

    $\left[\begin{array}{llll}1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{array}\right]$,

