# Math 290 ELEMENTARY LINEAR ALGEBRA <br> REVIEW OF LECTURES - VII 

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## Instructor: Yasuyuki Kachi

## Line \#: 25751.

§7. Matrix addition \& subtraction. Distributive Law.

## - Matrix addition, subtraction.

We studied matrix multiplication (in the $2 \times 2$ case). It will be no surprise to hear that matrix addition and subtraction make sense too. Today we do the $2 \times 2$ case. (Bear with me, I know you are so eager to see the larger size case. We will eventually cover it. But we take one step at a time.) In one line:
" Matrix addition and subtraction are done entry-wise. "
We officially define the matrix addition and subtraction as follows:

## Definition (Matrix addition/subtraction).

$$
\begin{array}{r}
\text { For } \begin{array}{r}
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right], \quad \text { and } \quad B=\left[\begin{array}{ll}
p & q \\
r & s
\end{array}\right], \quad \text { define } \\
\\
A+B=\left[\begin{array}{ll}
a+p & b+q \\
c+r & d+s
\end{array}\right], \\
A-B=\left[\begin{array}{ll}
a-p & b-q \\
c-r & d-s
\end{array}\right] .
\end{array}
\end{array}
$$

- In case you wonder: I somehow did matrix multiplication first, before addition and subtraction. Maybe that sounds a little unorthodox. The truth is, it doesn't matter, either way is completely viable. How to linearly align various topics is (primarily) dictated by their logical interdependence. Yet sometimes you randomly choose two topics and neither logically depends on the other. This is one of those instances. (There is something more to it: There is actually a way to define matrix addition via matrix multiplication. The only caveat will be it is $4 \times 4$ multiplication on which $2 \times 2$ addition is neatly framed. I will save details at this point.)
- Anyhow, let's take a look at some examples:

Example 1. For $A=\left[\begin{array}{ll}1 & 2 \\ 2 & 1\end{array}\right], \quad B=\left[\begin{array}{cc}-3 & -2 \\ 4 & 2\end{array}\right], \quad$ we have

$$
\begin{aligned}
& A+B=\left[\begin{array}{cc}
1+(-3) & 2+(-2) \\
2+4 & 1+2
\end{array}\right]=\left[\begin{array}{cc}
-2 & 0 \\
6 & 3
\end{array}\right] \\
& A-B=\left[\begin{array}{cc}
1-(-3) & 2-(-2) \\
2-4 & 1-2
\end{array}\right]=\left[\begin{array}{cc}
4 & 4 \\
-2 & -1
\end{array}\right] .
\end{aligned}
$$

Quiz 1. For $A$ and $B$ as in Example 1 above,
(1) do each of $\operatorname{det} A, \quad \operatorname{det} B, \quad \operatorname{det}(A+B) \quad$ and $\quad \operatorname{det}(A-B)$.
(2) True or false :

$$
\begin{equation*}
\operatorname{det} A+\operatorname{det} B \quad \text { equals } \quad \operatorname{det}(A+B) \tag{?}
\end{equation*}
$$

(3) True or false :

$$
\begin{equation*}
\operatorname{det} A-\operatorname{det} B \quad \text { equals } \quad \operatorname{det}(A-B) \tag{?}
\end{equation*}
$$

Solution. Let's do it together. First, as for (1):

$$
\begin{aligned}
\operatorname{det} A & =\left|\begin{array}{cc}
1 & 2 \\
2 & 1
\end{array}\right| \\
& =1 \cdot 1-2 \cdot 2=-3 \\
\operatorname{det} B & =\left|\begin{array}{cc}
-3 & -2 \\
4 & 2
\end{array}\right| \\
& =(-3) \cdot 2-(-2) \cdot 4=2
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{det}(A+B) & =\left|\begin{array}{cc}
-2 & 0 \\
6 & 3
\end{array}\right| \\
& =(-2) \cdot 3-0 \cdot 6=-6 \\
\operatorname{det}(A-B) & =\left|\begin{array}{cc}
4 & 4 \\
-2 & -1
\end{array}\right| \\
& =4 \cdot(-1)-4 \cdot(-2)=4
\end{aligned}
$$

So what do you see?

$$
\operatorname{det} A+\operatorname{det} B=-1 \quad \text { whereas } \quad \operatorname{det}(A+B)=-6
$$

so they are not equal. So, the answer for (2) is 'false'. Likewise

$$
\operatorname{det} A-\operatorname{det} B=-5 \quad \text { whereas } \quad \operatorname{det}(A-B)=4
$$

so they are not equal. So, the answer for (3) is 'false'.

- It is worthwhile to stress this aspect of the determinants:


## Warning 1.

" While the determinant operation is compatible with matrix multiplication, it is not compatible with matrix addition and subtraciton. "

Repeat: In general,

$$
\operatorname{det} A+\operatorname{det} B \quad \xlongequal{\text { and }} \quad \operatorname{det}(A+B) \quad \text { are not equal. }
$$

$$
\operatorname{det} A-\operatorname{det} B \quad \xlongequal{\text { and }} \quad \operatorname{det}(A-B) \quad \text { are not equal. }
$$

Exercise 1. For $A=\left[\begin{array}{cc}1 & 3 \\ 2 & -4\end{array}\right], \quad$ and $\quad B=\left[\begin{array}{ll}2 & 6 \\ 2 & 3\end{array}\right], \quad$ calculate
(1) $A+B$,
(2) $\operatorname{det}(A+B) \quad$ based on (1),
(3) $\operatorname{det} A+\operatorname{det} B$.

Do the answer for (2) and the answer for (3) coincide? Also calculate
(4) $A-B$,
(5) $\operatorname{det}(A-B) \quad$ based on (4),
(6) $\operatorname{det} A-\operatorname{det} B$.

Do the answer for (5) and the answer for (6) coincide?

- Having agreed with the above scope, let's do the next pop quiz.

Quiz 2. Recall $I=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$ (the identity matrix). So, for a scalar $\lambda$,

$$
\lambda I=\left[\begin{array}{ll}
\lambda & 0 \\
0 & \lambda
\end{array}\right] .
$$

(A strange choice of the letter indeed, but this is actually common.) Meanwhile, let

$$
A=\left[\begin{array}{ll}
3 & 1 \\
4 & 6
\end{array}\right]
$$

Do the subtraction $\lambda I-A$.

Solution. This is a piece of cake. Right? So, once again

$$
\lambda I=\left[\begin{array}{cc}
\lambda & 0 \\
0 & \lambda
\end{array}\right], \quad \text { and } \quad A=\left[\begin{array}{ll}
3 & 1 \\
4 & 6
\end{array}\right]
$$

So the answer for $\quad \lambda I-A$ is

$$
\lambda I-A=\left[\begin{array}{cc}
\lambda-3 & 0-1 \\
0-4 & \lambda-6
\end{array}\right]=\left[\begin{array}{cc}
\lambda-3 & -1 \\
-4 & \lambda-6
\end{array}\right]
$$

Good job.

- So we have just solved the quiz. For some reason, let's hang on a little bit longer. Stare at

$$
\lambda I-A=\left[\begin{array}{cc}
\lambda-3 & -1 \\
-4 & \lambda-6
\end{array}\right] .
$$

Quiz 3. (1) Calculate the determinant of this last matrix $\left[\begin{array}{cc}\lambda-3 & -1 \\ -4 & \lambda-6\end{array}\right]$.
(2) Solve the equation $\left|\begin{array}{cc}\lambda-3 & -1 \\ -4 & \lambda-6\end{array}\right|=0$.

Solution. As for part (1): Here we go:

$$
\begin{aligned}
\left|\begin{array}{cc}
\lambda-3 & -1 \\
-4 & \lambda-6
\end{array}\right| & =(\lambda-3)(\lambda-6)-(-1) \cdot(-4) \\
& =\left(\lambda^{2}-9 \lambda+18\right)-4 \\
& =\lambda^{2}-9 \lambda+14
\end{aligned}
$$

Don't be afraid that the entries of the matrix involve some letter. As for part (2): Factor the above outcome $\lambda^{2}-9 \lambda+14$ :

$$
\lambda^{2}-9 \lambda+14=(\lambda-2)(\lambda-7)
$$

So the roots for the equation

$$
\left|\begin{array}{cc}
\lambda-3 & -1 \\
-4 & \lambda-6
\end{array}\right|=0
$$

are

$$
\lambda=2, \quad \text { and } \quad \lambda=7
$$

- So we have nailed the two roots $\lambda=2$ and 7 of the equation $\operatorname{det}(\lambda I-A)=0$, for $A=\left[\begin{array}{ll}3 & 1 \\ 4 & 6\end{array}\right]$. For later purposes, let me throw some names (jargons):
- The two roots $\lambda=2, \lambda=7$ are called the eigenvalues of the matrix $A$.
- Spelling:


## eigenvalue.

The prefix 'eigen-' makes the word sound like German. Indeed, the word is coined from the German word 'Eigenwert'. Eigenvalues of a matrix $A$ nail the essence of $A$ as a "linear transform". We are going to spend a good deal of time on it. A couple more jargons closely associated with it:

- $\operatorname{det}(\lambda I-A)=\lambda^{2}-9 \lambda+14$ is called the characteristic polynomial of $A$. - $\lambda^{2}-9 \lambda+14=0$ is called the characteristic equation of $A$.

Terminology. Let $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right] . \quad$ (i) $\xlongequal{\text { the characteristic polynomial }}$


(iii) $\xlongequal{\text { the eigenvalues of } A \text { mean }} \xlongequal{\text { the roots of }} \operatorname{det}(\lambda I-A)=0, \quad$ or
$\xlongequal{\text { the same to say, }} \xlongequal{\text { the roots of }}\left|\begin{array}{cc}\lambda-a & -b \\ -c & \lambda-d\end{array}\right|=0$.

Warning 2. Don't ever simplify $\operatorname{det}(\lambda I-A)$ as $\operatorname{det}(\lambda I)-\operatorname{det} A$. That would be incorrect. Also, a formation like $\operatorname{det}(\lambda I)-\operatorname{det} A$ is of little significance.

Notation $\chi_{A}(\lambda)$ for the characteristic polynomial of $A$.
$\underline{\underline{\text { From now on, we denote the characteristic polynomial of }}} A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right] \xlongequal{\text { as }}$

$$
\chi_{A}(\lambda)
$$

So,

$$
\chi_{A}(\lambda)=\operatorname{det}(\lambda I-A)
$$

Or, the same to say

$$
\chi_{A}(\lambda)=\left|\begin{array}{cc}
\lambda-a & -b \\
-c & \lambda-d
\end{array}\right|
$$

Example 2. Find $\chi_{A}(\lambda)$ for $A=\left[\begin{array}{ll}8 & 3 \\ 6 & 5\end{array}\right]$. Find the eigenvalues of $A$.

- Well, this is a piece of cake. Here we go:

$$
\begin{aligned}
\chi_{A}(\lambda) & =\left|\begin{array}{cc}
\lambda-8 & -3 \\
-6 & \lambda-5
\end{array}\right| \\
& =(\lambda-8)(\lambda-5)-(-3) \cdot(-6) \\
& =\lambda^{2}-13 \lambda+22 .
\end{aligned}
$$

This is factored as

$$
\chi_{A}(\lambda)=(\lambda-2)(\lambda-11)
$$

so accordingly the eigenvalues of $A$ are found as $\lambda=2$, and $\lambda=11$.

Exercise 2. Find $\chi_{A}(\lambda)$. Find the eigenvalues of $A$.

$$
A=\left[\begin{array}{ll}
6 & 4  \tag{1}\\
6 & 1
\end{array}\right] . \quad(2) \quad A=\left[\begin{array}{cc}
\frac{1}{2} & \frac{-1}{2} \\
\frac{3}{2} & \frac{5}{2}
\end{array}\right]
$$

- Distributive Law. Now that we have defined
- addition, $\circ$ subtraction, and $\circ$ multiplication
of matrices, there arise some questions that need to be addressed pertaining to the situation when two or more of the operations are mixed. First, we often deal with 'expansions', like

$$
\begin{array}{lll}
A(B+C), & (B+C) D, & A(B+C) D, \\
A(B-C), & (B-C) D, & A(B-C) D,
\end{array}
$$

etc. Can we expand these like numbers? As in

$$
A(B+C)=A B+A C, \quad(B-C) D=B D-C D
$$

and like? The answer is, "yes".
Formula 1 (Distributive Laws; proof is Exercise 3 (page 10)). For

$$
A=\left[\begin{array}{ll}
a_{1} & b_{1} \\
c_{1} & d_{1}
\end{array}\right], \quad B=\left[\begin{array}{ll}
a_{2} & b_{2} \\
c_{2} & d_{2}
\end{array}\right], \quad C=\left[\begin{array}{ll}
a_{3} & b_{3} \\
c_{3} & d_{3}
\end{array}\right], \quad D=\left[\begin{array}{ll}
a_{4} & b_{4} \\
c_{4} & d_{4}
\end{array}\right]
$$

the following hold:

$$
\begin{align*}
A(B+C) & =A B+A C  \tag{1}\\
(B+C) D & =B D+C D  \tag{2}\\
A(B+C) D & =A B D+A C D  \tag{3}\\
A(B-C) & =A B-A C  \tag{4}\\
(B-C) D & =B D-C D  \tag{5}\\
A(B-C) D & =A B D-A C D \tag{6}
\end{align*}
$$

## - More on Distributive Laws.

There is always the next level. Let's take a look at the following:

$$
(A+B)(C+D)
$$

Quiz 4. Is the following the right way to do it?

$$
(A+B)(C+D)=A C+A D+B C+B D
$$

If so, then prove it. Use Formula 1 in the previous page.

Solution. Yes. Indeed. Let's temporarily call $C+D$ as $E$. So

$$
(A+B)(C+D)
$$

is $\quad(A+B) E . \quad$ According to part (2) of Formula 1, this is expanded as

$$
(A+B) E=A E+B E
$$

But since $E$ was the temporary name for $C+D$, so $A E+B E$ is

$$
A(C+D)+B(C+D)
$$

Now according to part (1) of Formula 1, this is expanded as

$$
A C+A D+B C+B D
$$

- In short, this quiz was the combination of part (1) and part (2) of Formula 1 in the previous page. Let's highlight:

Corollary 1. Let $A, B, C$ and $D$ be as above. Then

$$
(A+B)(C+D)=A C+A D+B C+B D
$$

## Exercise 3.

(a) Prove part (1) and part (2) of Formula 1 in page 1. As for part (1), physically calculate each of

$$
\begin{aligned}
& {\left[\begin{array}{ll}
a_{1} & b_{1} \\
c_{1} & d_{1}
\end{array}\right]\left(\left[\begin{array}{ll}
a_{2} & b_{2} \\
c_{2} & d_{2}
\end{array}\right]+\left[\begin{array}{ll}
a_{3} & b_{3} \\
c_{3} & d_{3}
\end{array}\right]\right), \quad \text { and }} \\
& {\left[\begin{array}{ll}
a_{1} & b_{1} \\
c_{1} & d_{1}
\end{array}\right]\left[\begin{array}{ll}
a_{2} & b_{2} \\
c_{2} & d_{2}
\end{array}\right]+\left[\begin{array}{ll}
a_{1} & b_{1} \\
c_{1} & d_{1}
\end{array}\right]\left[\begin{array}{ll}
a_{3} & b_{3} \\
c_{3} & d_{3}
\end{array}\right]}
\end{aligned}
$$

separately, and verify that they match. Part (2) is completely similar.
(b) Prove that part (3) of Formula 1 in page 1 is equivalent to parts (1-2) of the same formula. So, prove "(1) implies (2-3)" and "(2-3) imply (1)" both.

Example 3. Can we expand

$$
(A+B)(A+B) ?
$$

Sure. This is a special case of Corollary 1. Namely, the case $C=A$ and $D=B$ :

$$
(A+B)(A+B)=A A+A B+B A+B B
$$

But can't we rewrite this as

$$
(A+B)^{2}=A^{2}+A B+B A+B^{2} ?
$$

Sure. But then we should be careful. Don't try to simplify it further. $A B$ and $B A$ are usually not the same. So you might be tempted to combine them and throw $2 A B$ for $A B+B A$, but you can't. Let's highlight:

Corollary 2. For $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right], \quad$ and $\quad B=\left[\begin{array}{ll}p & q \\ r & s\end{array}\right]$,

$$
(A+B)^{2}=A^{2}+A B+B A+B^{2}
$$

Example 4. Next, can we expand

$$
(A+B)(A-B) ?
$$

Sure. But prior to solving that, we will be benefited by the following:

Formula 2 (Distributive \& Associative Laws Involving Scalars - I). For

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right], \quad B=\left[\begin{array}{ll}
p & q \\
r & s
\end{array}\right]
$$

the following hold:

$$
\begin{align*}
-(A+B) & =-A-B  \tag{1}\\
(-A) B & =-(A B)  \tag{2}\\
A(-B) & =-(A B) \tag{3}
\end{align*}
$$

More generally, for a scalar $t$,

$$
\begin{align*}
t(A+B) & =t A+t B  \tag{4}\\
(t A) B & =t(A B)  \tag{5}\\
A(t B) & =t(A B) \tag{6}
\end{align*}
$$

(Clearly part (1-3) are the case $t=-1$ of part (4-6).)

Example 4 resumed. Back to

$$
(A+B)(A-B)
$$

how do we go about? Yes. Let's temporarily call $A+B$ as $E$. So this is $E(A-B)$. According to part (4) of Formula 1 in page 8, this is expanded as

$$
E(A-B)=E A-E B
$$

But since $E$ was the temporary name for $A+B$, so this is

$$
(A+B) A-(A+B) B
$$

Now this is expanded as

$$
A A+B A-(A B+B B)
$$

that is,

$$
A A+B A+(-(A B+B B))
$$

that is,

$$
A A+B A+(-A B-B B)
$$

(where we have used part (1) of Formula 2 in page 11), that is,

$$
A A+B A-A B-B B
$$

Rewrite $A A$ as $A^{2}$, also $B B$ as $B^{2}$, so we arrive at

$$
(A+B)(A-B)=A^{2}+B A-A B-B^{2}
$$

Here, again, we should be careful. There is no room for further simplification. $A B$ and $B A$ are usually not the same. You might be tempted to cancel them, but you can't. Let's highlight what we've got:

Corollary 3. For $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right], \quad$ and $\quad B=\left[\begin{array}{ll}p & q \\ r & s\end{array}\right]$,

$$
(A+B)(A-B)=A^{2}+B A-A B-B^{2}
$$

Example 5.

$$
\begin{aligned}
A(I-B) C & =A I C-A B C \\
& =A C-A B C
\end{aligned}
$$

Example 6a. Let's tweak it:

$$
\begin{aligned}
A(2 I-B) C & =A(2 I) C-A B C \\
& =2 A C-A B C
\end{aligned}
$$

Example 6b. In the above, suppose $C=A^{-1}$ :

$$
\begin{aligned}
A(2 I-B) A^{-1} & =2 A A^{-1}-A B A^{-1} \\
& =2 I-A B A^{-1}
\end{aligned}
$$

Example 7.

$$
\begin{aligned}
\left(A B A^{-1}\right)^{2} & =A B A^{-1} A B A^{-1} \\
& =A B I B A^{-1} \\
& =A B B A^{-1} \\
& =A B^{2} A^{-1}
\end{aligned}
$$

Example 8a. Let's add $4 I$ :

$$
4 I+\left(A B A^{-1}\right)^{2}=4 I+A B^{2} A^{-1}
$$

Example 8b. Meanwhile

$$
\begin{aligned}
A\left(4 I+B^{2}\right) A^{-1} & =A(4 I) A^{-1}+A B^{2} A^{-1} \\
& =4 A I A^{-1}+A B^{2} A^{-1} \\
& =4 I+A B^{2} A^{-1}
\end{aligned}
$$

Example 8c. If you combine the results of Examples 8a and 8b, you see

$$
4 I+\left(A B A^{-1}\right)^{2}=A\left(4 I+B^{2}\right) A^{-1}
$$

A little shift of a paradigm: Think of $4 I+B^{2}$ as $f(B)$, where $f(x)=4+x^{2}$. Likewise, think of $4 I+\left(A B A^{-1}\right)^{2}$ as $f\left(A B A^{-1}\right)$. Then (\#) is rewritten as (\#) ${ }^{\prime}$

$$
f\left(A B A^{-1}\right)=A f(B) A^{-1}
$$

More on this later. Finally, to wrap it up, let me throw

Formula 3 (Distributive \& Associative Laws Involving Scalars - II). For

$$
A=\left[\begin{array}{ll}
a_{1} & b_{1} \\
c_{1} & d_{1}
\end{array}\right], \quad B=\left[\begin{array}{ll}
a_{2} & b_{2} \\
c_{2} & d_{2}
\end{array}\right], \quad C=\left[\begin{array}{ll}
a_{3} & b_{3} \\
c_{3} & d_{3}
\end{array}\right], \quad D=\left[\begin{array}{ll}
a_{4} & b_{4} \\
c_{4} & d_{4}
\end{array}\right],
$$

the following hold:

$$
\begin{align*}
& (-A)(B+C)=-A B-A C  \tag{1}\\
& (B+C)(-D)=-B D-C D \tag{2}
\end{align*}
$$

More generally, for a scalar $t$,

$$
\begin{align*}
& (t A)(B+C)=t A B+t A C  \tag{3}\\
& (B+C)(t D)=t B D+t C D \tag{4}
\end{align*}
$$

(Clearly part (1-2) are the case $t=-1$ of part (3-4).) Also, for scalars $t$ and $u$,

$$
\begin{align*}
A(t B+u C) & =t A B+u A C  \tag{5}\\
(t B+u C) D & =t B D+u C D \tag{6}
\end{align*}
$$

(Clearly part (1-2), and part (4-5), of Formula 1 in page 8 are the case $t=1$, $u=1$; and the case $t=1, u=-1$, of part (5-6) above, respectively.)

