Math 290 ELEMENTARY LINEAR ALGEBRA

REVIEW OF LECTURES – VI

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Instructor: Yasuyuki Kachi

Line #: 25751.

§6. Product Formula. Associativity Law.

As promised, we cover two items today —

- ◦ [**a**] Product Formula.
- ◦ [**b**] Associativity Law.

• [a] **Product Formula.**

Up until now our lectures have been centered around two things:

(i) Determinants (introduced in "Review of Lectures – II"); and

(ii) Matrix multiplications (introduced in "Review of Lectures – IV").

These two concepts emerged from two different sources. So far we haven't seen them intermingled except the formation of the inverse matrix A^{-1} somehow involves a determinant ((i)), whereas the inverse matrix is nothing but the matrix analog of the reciprocal of numbers, so technically it can also be within the framework of matrix multiplications ((ii)). Still that's a tenuous connection between (i) and (ii), so to speak. So you might have had an impression this class merely offers some mishmash of ideas in an omnibus style. Now, 'Product Formula' (see page 4) postulates, on the contrary, that those two notions (i) and (ii) totally go together, they "couldn't be more compatible". And that's just a tiny snapshot of how this class goes — so far you guys are learning a lot of miscellaneous stuffs, but count that ultimately those pieces will crystallize into one integrated mass of knowledge. What makes such an integration possible is this vital concept in linear algebra, on which everything else ultimately hinges: "Abstract vector spaces", which is also naturally accompanied by the notion of "linearity". You'll hear more on that once the second-half of the semester kicks off. That's what we are aiming at.

So, 'Product Formula' exemplifies the mutual *affinity* between (i) determinants, and (ii) matrix multiplications. Let me use some examples to illustrate it:

Example 1 (that is in sync with Product Formula). Let

$$A = \begin{bmatrix} 2 & 1 \\ -4 & 7 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 6 & -3 \\ 5 & -1 \end{bmatrix}.$$

(These two are just randomly picked.) Let's calculate $\det A$ and $\det B$:

$$\begin{aligned} \det A &= \begin{vmatrix} 2 & 1 \\ -4 & 7 \end{vmatrix} \\ &= 2 \cdot 7 - 1 \cdot (-4) = 18, \end{aligned}$$

$$\begin{aligned} \det B &= \begin{vmatrix} 6 & -3 \\ 5 & -1 \end{vmatrix} \\ &= 6 \cdot (-1) - (-3) \cdot 5 = 9. \end{aligned}$$

Mmm. So far so good. But then, independently of those, why don't we calculate AB (not the determinant yet, just A times B):

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 1 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 6 + 1 \cdot 5 & 2 \cdot (-3) + 1 \cdot (-1) \\ (-4) \cdot 6 + 7 \cdot 5 & (-4) \cdot (-3) + 7 \cdot (-1) \end{bmatrix} \\ &= \begin{bmatrix} 17 & -7 \\ 11 & 5 \end{bmatrix}. \end{aligned}$$

Do you know where I am going? Yes. I want you to now calculate the determinant of this last one $\begin{bmatrix} 17 & -7 \\ 11 & 5 \end{bmatrix}$, and see what happens. Voilà:

$$\det\left(AB\right) = 17 \cdot 5 - (-7) \cdot 11 = 162.$$

To summarize,

$$\det A = 18, \qquad \det B = 9, \qquad \det\left(A\,B\right) = 162.$$

Realize $18 \cdot 9 = 162$.

Example 2 (that is in sync with Product Formula). Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}.$$

(Once again these matrices are random choices.) Agree

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \\ &= 1 \cdot (-1) - 3 \cdot 2 = -7, \end{aligned}$$

$$\begin{aligned} \det B &= \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} \\ &= 2 \cdot 5 - 1 \cdot 4 = 6. \end{aligned}$$

Independently of these,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 + 3 \cdot 4 & 1 \cdot 1 + 3 \cdot 5 \\ 2 \cdot 2 + (-1) \cdot 4 & 2 \cdot 1 + (-1) \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 16 \\ 0 & -3 \end{bmatrix}, \end{aligned}$$

so

$$\det (AB) = 14 \cdot (-3) - 16 \cdot 0 = -42.$$

To summarize:

$$\det A = -7, \qquad \det B = 6, \qquad \det (AB) = -42.$$

Realize $(-7) \cdot 6 = -42$.

- So in both examples (Example 1 and Example 2),

$$\det\Big(AB\Big) = \Big(\det A\Big)\Big(\det B\Big).$$

Question. Is that a coincidence?

(You knew this question coming, right?)

Answer. No, that is not a coincidence. The same is actually true for any

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

(You knew this answer coming too.)

- Below is the first highlight of the day:

Formula 1 (Product Formula for 2×2).

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, we have

$$\boxed{\det\Big(AB\Big) = \Big(\det A\Big)\Big(\det B\Big).}$$

Okay, that was just that. So, let's move on. What do you think will be next? Or what?

— Wait.. We are not done yet. We still need to prove Product Formula. Don't say "Oh, no. Again?" If you say we have already tested the validity of this formula in two different examples, so you can trust that the same formula must always be true, so you want to waste no time and get to the next topic. If you say so, I say "sorry, but you are mistaken".

True, the above two examples are both in sync with what the formula says. But that might be by sheer accident. Even if you try out with one hundred, one thousand, or one million, examples, all of which are in sync with what Product Formula says, that's not enough. Because there are infinitely many matrices A and B. So, you really have to prove the statement using a general pair of matrices A and B :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

where a, b, c, d, p, q, r and s are arbitrary.

• All right, I hope I have convinced you. How should you go about proving it, though? Let's dissect. First agree

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \det B = \begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - qr.$$

Meanwhile, agree

$$AB = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}.$$

So

$$\begin{aligned} \det\Big(AB\Big) &= \begin{vmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{vmatrix} \\ &= \Big(ap + br\Big)\Big(cq + ds\Big) - \Big(aq + bs\Big)\Big(cp + dr\Big). \end{aligned}$$

Bases on these, let's agree that the content of Product Formula $\big($Formula 1$\big)$ in the previous page is a mere paraphrase of the following:

Formula 1′ (**Product Formula,** **spelt-out version,** $\mathbf{2 \times 2}$).

$$(*) \qquad \boxed{\begin{aligned} &\Big(ap + br\Big)\Big(cq + ds\Big) - \Big(aq + bs\Big)\Big(cp + dr\Big) \\ &\qquad\qquad = \Big(ad - bc\Big)\Big(ps - qr\Big). \end{aligned}}$$

This is what I call "the spelt-out version" of Formula 1, meaning:

" *In order to prove* Formula 1, *it suffices to prove* $(*)$ (Formula 1′). "

Good news: Verifying $(*)$ is straightforward (mundane exercise):

Proof of $(*)$ (Formula 1′).

The left-hand side of $(*)$

$$\begin{aligned} &= \Big(ap + br\Big)\Big(cq + ds\Big) - \Big(aq + bs\Big)\Big(cp + dr\Big) \\ &= \Big(apcq + apds + brcq + brds\Big) - \Big(aqcp + aqdr + bscp + bsdr\Big) \\ &= \Big(acpq + adps + bcqr + bdrs\Big) - \Big(acpq + adqr + bcps + bdrs\Big) \\ &= adps + bcqr - adqr - bcps. \end{aligned}$$

The right-hand side of $(*)$

$$\begin{aligned} &= \Big(ad - bc\Big)\Big(ps - qr\Big) \\ &= adps - adqr - bcps + bcqr \\ &= adps + bcqr - adqr - bcps. \end{aligned}$$

The above calculations show that the two sides of $(*)$ are equal. $\square$

Exercise 1. For $A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$, and $B = \begin{bmatrix} 6 & 5 \\ 8 & 3 \end{bmatrix}$, calculate

(1) $\det A$, (2) $\det B$, (3) $\Big(\det A\Big)\Big(\det B\Big)$ based on (1–2),

(4) AB, and (5) $\det\Big(AB\Big)$ based on (4).

Confirm that the answer for (3) and the answer for (5) coincide.

• We have just proved Product Formula $\big($Formula 1 on page 4$\big)$ via Formula 1′ $\big($on page 6$\big)$. You would sense that this is one of those "*ad nauseum*", something you are compelled to memorize. In fact, the level of sophistication is subpar, at best. The truth is, although this formula itself appears to be rather rudimentary, it has larger size counterparts, and those will not be as rudimentary. You may not believe me. So, let's just take a quick peek at how each of the 3×3 and the 4×4 counterparts looks like $\big($below$\big)$. How do you see them? Do they strike you as trivial?

Product Formula (Spelt-out version, $\mathbf{3 \times 3}$).

$$
\begin{aligned}
&\Big(a_1p_1 + a_2p_2 + a_3p_3\Big)\Big(b_1q_1 + b_2q_2 + b_3q_3\Big)\Big(c_1r_1 + c_2r_2 + c_3r_3\Big) \\
&+ \Big(a_1q_1 + a_2q_2 + a_3q_3\Big)\Big(b_1r_1 + b_2r_2 + b_3r_3\Big)\Big(c_1p_1 + c_2p_2 + c_3p_3\Big) \\
&+ \Big(a_1r_1 + a_2r_2 + a_3r_3\Big)\Big(b_1p_1 + b_2p_2 + b_3p_3\Big)\Big(c_1q_1 + c_2q_2 + c_3q_3\Big) \\
&- \Big(a_1r_1 + a_2r_2 + a_3r_3\Big)\Big(b_1q_1 + b_2q_2 + b_3q_3\Big)\Big(c_1p_1 + c_2p_2 + c_3p_3\Big) \\
&- \Big(a_1p_1 + a_2p_2 + a_3p_3\Big)\Big(b_1r_1 + b_2r_2 + b_3r_3\Big)\Big(c_1q_1 + c_2q_2 + c_3q_3\Big) \\
&- \Big(a_1q_1 + a_2q_2 + a_3q_3\Big)\Big(b_1p_1 + b_2p_2 + b_3p_3\Big)\Big(c_1r_1 + c_2r_2 + c_3r_3\Big) \\
&= \Big(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3\Big) \\
&\quad \cdot \Big(p_1q_2r_3 + p_2q_3r_1 + p_3q_1r_2 - p_3q_2r_1 - p_1q_3r_2 - p_2q_1r_3\Big) .
\end{aligned}
$$

Product Formula (Spelt-out version, 4×4).

$$
\begin{aligned}
&(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&+(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
&+(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&+(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&+(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&+(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&+(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&+(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&+(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
&+(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
&+(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&+(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&-(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&-(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
&-(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&-(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&-(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&-(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&-(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
&-(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&-(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&-(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&-(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&-(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \Big(a_1b_2c_3d_4+a_1b_3c_4d_2+a_1b_4c_2d_3+a_2b_1c_4d_3+a_2b_4c_3d_1+a_2b_3c_1d_4\\
&+a_3b_1c_2d_4+a_3b_2c_4d_1+a_3b_4c_1d_2+a_4b_1c_3d_2+a_4b_3c_2d_1+a_4b_2c_1d_3\\
&-a_1b_2c_4d_3-a_1b_4c_3d_2-a_1b_3c_2d_4-a_2b_1c_3d_4-a_2b_3c_4d_1-a_2b_4c_1d_3\\
&-a_3b_1c_4d_2-a_3b_4c_2d_1-a_3b_2c_1d_4-a_4b_1c_2d_3-a_4b_2c_3d_1-a_4b_3c_1d_2 \Big)\\
\cdot \quad & \Big(p_1q_2r_3s_4+p_1q_3r_4s_2+p_1q_4r_2s_3+p_2q_1r_4s_3+p_2q_4r_3s_1+p_2q_3r_1s_4\\
&+p_3q_1r_2s_4+p_3q_2r_4s_1+p_3q_4r_1s_2+p_4q_1r_3s_2+p_4q_3r_2s_1+p_4q_2r_1s_3\\
&-p_1q_2r_4s_3-p_1q_4r_3s_2-p_1q_3r_2s_4-p_2q_1r_3s_4-p_2q_3r_4s_1-p_2q_4r_1s_3\\
&-p_3q_1r_4s_2-p_3q_4r_2s_1y-p_3q_2r_1s_4-p_4q_1r_2s_3-p_4q_2r_3s_1-p_4q_3r_1s_2 \Big).
\end{aligned}
$$

Now, suppose I asked you to *prove* these, you probably cannot think of any immediate way to go about. If you say your computer can handle it, I say this: The above is only up to 4×4. The same formula for 5×5, 6×6, 7×7, $\cdots$ exist. The expansion of the left-hand side, and the right-hand side, of the $n \times n$ counterpart formula, before cancellations, involve the following number of terms:

$$n^n \cdot n!, \qquad \text{and} \qquad \Big(n!\Big)^2,$$

respectively. These numbers for $n = 3, 4, 5, 6, 7, 8$ come out as

$$\begin{array}{lcrr}
n = 3 & \Longrightarrow & 162, & 36. \\
n = 4 & \Longrightarrow & 6144, & 576. \\
n = 5 & \Longrightarrow & 375000, & 14400. \\
n = 6 & \Longrightarrow & 33592320, & 518400. \\
n = 7 & \Longrightarrow & 4150656720, & 25401600. \\
n = 8 & \Longrightarrow & 676457349120, & 1625702400. \\
\vdots & & \vdots & \vdots
\end{array}$$

These are the numbers of terms your computer is supposed to deal with. These numbers grow exponentially as n grows. As you can read off from the above table, already for $n = 8$ the number $n^n \cdot n!$ is a 12-digit number $\big($an order of trillion$\big)$. For $n = 100$ the corresponding number $n^n \cdot n!$ is a 358-digit number. Sooner or later it will go above your computer's capability. Now, be that as it may, any mathematical software you've heard of actually knows the Product Formula in any size n. The next thing I say is important: That's because the formula is known to be true, by humans, because human mathematicians have logically proved it, in an *ex machina* way. Then whoever came up with that software $\big($or whoever is in charge of updating that software$\big)$ borrowed that knowledge and installed the formula on the software. The computer itself does not have enough intelligence to generate that proof of the formula. How to prove the formula, in an *ex machina* way, and things of this nature, are what you are going to learn in this class: You are going to work on things computers cannot replace. So far we haven't even defined the determinants for matrices larger than 3×3, or matrix multiplication for matrices larger than 2×2. We need to do that first. Then how to prove Product Formula for $n \times n$ is a whole different story altogether. That's coming up. For the rest of today we switch gears.

- **[b] Associativity Law.**

In matrix arithmetic, we often deal with the situation where <u>three</u> matrices are involved. Suppose we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad C = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

We are sometimes compelled to form their product ABC by necessity. We have already encountered such situation in "Review of Lectures – V"; page 10 — what we've seen there is more like $A^{-1}AB$, but this falls into the template of "three matrices being multiplied together". So let's talk about ABC, which is more general.

As innocuous as it seems, if you stop and think twice, we have to worry about the following: There are two obvious ways to calculate it:

(i) Calculate ABC as $\Big(AB\Big)C$. (ii) Calculate ABC as $A\Big(BC\Big)$.

A natural question here is, whether these two match. This is something we need to analyze. The answer is, that is indeed the case. In fact, here is the proof:

Proof of $\boldsymbol{\Big(AB\Big)C = A\Big(BC\Big)}$.

$$\begin{aligned} \Big(AB\Big)C &= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \right) \begin{bmatrix} x & y \\ z & w \end{bmatrix} \\ &= \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \\ &= \begin{bmatrix} apx + brx + aqz + bsz & apy + bry + aqw + bsw \\ cpx + drx + cqz + dsz & cpy + dry + cqw + dsw \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} A\Big(BC\Big) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \right) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} px + qz & py + qw \\ rx + sz & ry + sw \end{bmatrix} \\ &= \begin{bmatrix} apx + aqz + brx + bsz & apy + aqw + bry + bsw \\ cpx + cqz + drx + dsz & cpy + cqw + dry + dsw \end{bmatrix}. \end{aligned}$$

If you look at these, you see that $\Big(AB\Big)C$ and $A\Big(BC\Big)$ indeed coincide. $\square$

- To highlight the result:

> **Formula 2 (Associativity Law).** For
>
> $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \qquad C = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$
>
> we have
>
> $$\boxed{\Big(AB\Big)C = A\Big(BC\Big).}$$

- Proof of Formula 2 is already given in the previous page, so we don't need to repeat that. Now, what does Formula 2 above entail? Yes, when it comes to ABC, we don't ever have to worry about placing parenthesis either over the AB part, or over the BC part. This is just like

$$2 \cdot 3 \cdot 5.$$

You immediately say 30 is the answer. I bet you probably did it this way: First $2 \cdot 3 = 6$, and then $6 \cdot 5 = 30$. But like I said "probably". Indeed, some of you might have done it the following way: First $3 \cdot 5 = 15$, and then $2 \cdot 15 = 30$. But we all know that, either way you'll get the same answer. Below is the mathematically precise way to compile what I just said $\big($pay attention to the parentheses$\big)$:

$$\Big(2 \cdot 3\Big) \cdot 5 = 2 \cdot \Big(3 \cdot 5\Big).$$

More generally, if a, b and c are numbers $\big($real numbers, to be precise$\big)$, then

$$\Big(a \cdot b\Big) \cdot c = a \cdot \Big(b \cdot c\Big).$$

So, you would say Formula 2 above *merely* says that the same is true for matrices. Well, that's true. So then you might say Formula 2 is not worthy to isolate because that's no surprise. Well, I have at least two ways to retort. One is the following $\big($this is actually the second time you hear this$\big)$: When you generalize something from numbers to matrices, you are going to lose some of the properties. In general $AB \neq BA$ for matrices A and B. Needless to say, $ab = ba$ for numbers a and b. <u>So you need to keep track of both (a) those properties that are carried over from numbers to matrices, and (b) those that aren't.</u>

Another: there is actually a number system wherein

$$\Big(a \cdot b\Big) \cdot c \;\neq\; a \cdot \Big(b \cdot c\Big).$$

It's just that we haven't seen it yet in this class. Numbers in such number system are called <u>octonion numbers</u> (octonions), or sometimes called <u>Cayley numbers</u>. But you don't really have to know what that is. In order to understand octonion numbers you need to understand <u>quaternion numbers</u> (quaternions) first, indeed, the former is a generalization of the latter. If you are interested, we can chat about it on the sideline. Not here, not right now.

Definition (triple product). Keeping Formula 2 in mind, we define

$$ABC$$

as either $A\Big(BC\Big)$, or equivalently, $\Big(AB\Big)C$.

- **Multiplications of four or more matrices.** Next, for

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}, \quad D = \begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix},$$

there are apparently five different ways to calculate $ABCD$:

(i) Calculate $ABCD$ as $\Big(\big(AB\big)C\Big)D$.

(ii) Calculate $ABCD$ as $\Big(A\big(BC\big)\Big)D$.

(iii) Calculate $ABCD$ as $A\Big(\big(BC\big)D\Big)$.

(iv) Calculate $ABCD$ as $A\Big(B\big(CD\big)\Big)$.

(v) Calculate $ABCD$ as $\Big(AB\Big)\Big(CD\Big)$.

Do you see that all these five (i–v) coincide?

Exercise 2. Explain why (i–v) all coincide.

$\left[\underline{\textbf{Hint for Exercise 2}}\right]$**:** First explain why (i) and (ii) are the same. For that matter, it suffices to say (i) and (ii) are both $\Big(ABC\Big)D$. Next, explain why (iii) and (iv) are the same $\big($same logic$\big)$. Next, explain why (i) and (v) are the same $\big($set $AB = E\big)$. Finally, explain why (iv) and (v) are the same $\big($set $CD = F\big)$.

It is worth highlighting the content of Exercise 2 $\big($below$\big)$:

Corollary. Let A, B, C, D be as above. Then

$$\begin{aligned}\Big((AB)\,C\Big)\,D \;&=\; \Big(A\,(BC)\Big)\,D \;=\; A\Big((BC)\,D\Big)\\ &=\; A\Big(B\,(CD)\Big) \;=\; \Big(AB\Big)\Big(CD\Big).\end{aligned}$$

Definition. Keeping Corollary above in mind, we define $ABCD$ as the five mutually equal matrices:

$$\begin{aligned}ABCD \;=\; \Big((AB)\,C\Big)\,D \;&=\; \Big(A\,(BC)\Big)\,D \;=\; A\Big((BC)\,D\Big)\\ &=\; A\Big(B\,(CD)\Big) \;=\; \Big(AB\Big)\Big(CD\Big).\end{aligned}$$

• **Consecutive product.** We may extend the above idea, and may define a consecutive product for an arbitrary number of matrices.

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}, \quad \cdots, \quad A_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}.$$

Define the product $A_1\,A_2\,A_3\cdots A_{k-1}A_k$ as

$$\begin{aligned}A_1\,A_2\,A_3\cdots A_{k-1}A_k \;&=\; \Bigg(\Big(\big(\,\cdot\cdot\,((A_1\,A_2)\,A_3)\,\cdots\,\big)A_{k-2}\Big)\,A_{k-1}\Bigg)\,A_k\\ &=\; A_1\Bigg(A_2\,\Big(A_3\,\big(\,\cdots\,(A_{k-2}\,(A_{k-1}\,A_k))\,\cdot\cdot\,\big)\Big)\Bigg).\end{aligned}$$

Exercise 3. Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Calculate

(1) AB. (2) BC. (3) CD.
(4) ABC. (5) BCD. (6) $ABCD$.

- **Powers.** In the product

$$A_1 A_2 A_3 \ \cdots \ A_k,$$

suppose $A_1, A_2, A_3, \cdots, A_k$ are mutually identical, call it A. Then we might as well just write it as A^k. In other words:

Definition. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, consider

$$\begin{aligned} A^1 &= A, \\ A^2 &= AA, \\ A^3 &= AAA, \\ A^4 &= AAAA, \\ A^5 &= AAAAA, \\ &\ \vdots \end{aligned}$$

- So

$$A^k = \underbrace{A\ A\ A\ \cdots\ A}_{k}.$$

Example 3. Remember that $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ satisfies

$$IA = A$$

no matter what $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is. So, nothing stops us from setting $A = I$, and that way we get

$$II = I.$$

In other words,

$$I^2 = I.$$

From this we also get

$$\begin{aligned} I\,I^2 &= I\,I \\ &= I. \end{aligned}$$

In other words, $I^3 = I$. From this we also get

$$\begin{aligned} I\,I^3 &= I\,I \\ &= I. \end{aligned}$$

In other words, $I^4 = I$. And this goes on and on.

So, in short, for the identity matrix I, we have

$$I^k = I, \quad \text{for} \quad k = 1, 2, 3, \cdots .$$

Stated in other words,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{for} \quad k = 1, 2, 3, \cdots .$$

• One last topic to wrap up today's class. In the matrix multiplication formation, suppose some constant is being multiplied to each of the matrices, like

$$\Big(tA\Big)\Big(uB\Big).$$

Then we can pull those constants to the left. Below is the precise statement:

Formula 3. Let t and u be scalars. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Then

$$\Big(tA\Big)\Big(uB\Big) = (tu)\Big(AB\Big).$$

Corollary. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and t a scalar. Then for $k = 1, 2, 3, \cdots$,

$$\Big(t\,A\Big)^k = t^k\,A^k.$$

Example 4. In Corollary above, if you set $A = I$, then

$$A = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \qquad \Longrightarrow \qquad A^k = \begin{bmatrix} t^k & 0 \\ 0 & t^k \end{bmatrix}.$$

The following is a generalization of Example 4:

Example 5. Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then

$(*)_k$
$$A^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}.$$

Exercise 4. Prove Example 5, via mathematical induction. Practically, do (i) and (ii) below:

(i) Prove that $(*)_1$ $\big(= (*)_k$ for $k = 1\big)$ is true.

(ii) Assume that $(*)_k$ is true, and with that assumption prove that $(*)_{k+1}$ is true.

Exercise 5. For $k = 1, 2, 3, \cdots,$ find

(1) $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^k.$ (2) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^k.$ (3) $\begin{bmatrix} 2 & a \\ 0 & 2 \end{bmatrix}^k.$

$\left[\underline{\textbf{Hint for Exercise 5 (2)}}\right]$**:** First verify

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}.$$

Use this fact wisely.

Exercise 6. (1) True or False. $\left(AB\right)^2 = A^2B^2.$

(2) Suppose $AB = BA.$ True or False. $\left(AB\right)^2 = A^2B^2.$