# Math 290 ELEMENTARY LINEAR ALGEBRA REVIEW OF LECTURES - V 

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§5. Matrix arithmetic - III. Identity matrix.

- Today we start with one specific matrix that plays an important role:

Definition. $\quad\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$ is called the $(2 \times 2) \xrightarrow{\text { identity matrix } . ~ W e ~ r e s e r v e ~ t h e ~}$ letter " $I$ " for this matrix:

$$
I=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]
$$

- Okay, that was out of the blue. What's so special about it? Why does this matrix $\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$ deserve a special treatment (a designated name, and a letter)? The next example explains it:

Example 1. Let

$$
A=\left[\begin{array}{cc}
2 & 4 \\
7 & -3
\end{array}\right]
$$

(I just randomly created this $A$.) Not for nothing, let's calculate $I A$ (matrix multiplication, from our last lecture): Here we go:

$$
\begin{aligned}
I A & =\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{cc}
2 & 4 \\
7 & -3
\end{array}\right] \\
& =\left[\begin{array}{ll}
1 \cdot 2+0 \cdot 7 & 1 \cdot 4+0 \cdot(-3) \\
0 \cdot 2+1 \cdot 7 & 0 \cdot 4+1 \cdot(-3)
\end{array}\right]=\left[\begin{array}{cc}
2 & 4 \\
7 & -3
\end{array}\right] .
\end{aligned}
$$

All right. What do you notice? Isn't the final outcome just $A$ ? So,

$$
I A=A
$$

for this particular $A$ : $A=\left[\begin{array}{cc}2 & 4 \\ 7 & -3\end{array}\right]$. Fine. But how about $A I$ ? Remember, for two matrices $A$ and $B, A B$ and $B A$ may or may not coincide, depending on $A$ and $B$ (see "Review of Lectures - IV", page 11). So we shouldn't prematurely conclude that $A I$ equals $A$, just from knowing that $I A$ equals $A$. Not yet. I'm still not saying that it is one way or the other. So, let's check:

$$
\begin{aligned}
A I & =\left[\begin{array}{cc}
2 & 4 \\
7 & -3
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right] \\
& =\left[\begin{array}{cc}
2 \cdot 1+4 \cdot 0 & 2 \cdot 0+4 \cdot 1 \\
7 \cdot 1+(-3) \cdot 0 & 7 \cdot 0+(-3) \cdot 1
\end{array}\right]=\left[\begin{array}{cc}
2 & 4 \\
7 & -3
\end{array}\right] .
\end{aligned}
$$

Okay. The outcome is indeed $A$. - We kind of expected it, though.
So,

$$
A I=A
$$

for the same $A$ : $A=\left[\begin{array}{cc}2 & 4 \\ 7 & -3\end{array}\right] . \quad$ In sum, we have verified:

$$
A=\left[\begin{array}{cc}
2 & 4 \\
7 & -3
\end{array}\right] \quad \Longrightarrow \quad I A=A, \quad A I=A
$$

- Now, from this you naturally suspect that the same is true not just for one $A$, $\xlongequal{\text { but for all } A \text {, as long as } A \text { is a } 2 \times 2 \text { matrix. If you want to know the answer right }}$ way, here it is: 'Yes indeed'. How ever innocuous, that fact is worthy to highlight:

Fact 1. For $I=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right] \quad$ and $\quad A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right], \quad$ we have

$$
I A=A, \quad \text { and } \quad A I=A
$$

- We need to give a proof of this statement. You might say we have already tested the validity of this statement using one concrete $A$ : $A=\left[\begin{array}{cc}2 & 4 \\ 7 & -3\end{array}\right]$, and that's enough (to claim that the statement is valid). If you say so, I say "well, not really". The statement might be true for one $A$ by sheer coincidence. The same statement
 $\underline{\underline{\text { that is, }}}$

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

$\xlongequal{\text { where } a, b, c \text { and } d \text { are arbitrary. So, here we go: }}$

Proof. Let's do $I A$ and $A I$ for

$$
I=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right], \quad \text { and } \quad A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

Here we go:

$$
\begin{aligned}
I A & =\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \\
& =\left[\begin{array}{ll}
1 \cdot a+0 \cdot c & 1 \cdot b+0 \cdot d \\
0 \cdot a+1 \cdot c & 0 \cdot b+1 \cdot d
\end{array}\right] \\
& =\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]=A, \\
A I & =\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] \\
& =\left[\begin{array}{ll}
a \cdot 1+b \cdot 0 & a \cdot 0+b \cdot 1 \\
c \cdot 1+d \cdot 0 & c \cdot 0+d \cdot 1
\end{array}\right] \\
& =\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]=A .
\end{aligned}
$$

- Since this is important, let me highlight it one more time:

Fact 1. For $I=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right] \quad$ and $\quad A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right], \quad$ we have

$$
I A=A, \quad \text { and } \quad A I=A
$$

A good analogy:
"In the context of matrix multiplications, the identity matrix ' $I$ '
serves the same role as ' 1 ' (the number) does in the usual number
multiplications. We always have
$(*) \quad 1 a=a, \quad$ and $\quad a 1=a$
for any number a. (Right?) In the same token,
(\#) $\quad I A=A, \quad$ and $\quad A I=A$
for any matrix $A$. These two, $(*)$ and (\#), are entirely parallel."

- Now, let me quiz you.

Quiz. Let $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$, and suppose $A^{-1}$ exists.

$$
A A^{-1}=? \quad A^{-1} A=?
$$

- If you say both equal $I$, I'd say you are very smart: You are indeed correct. Doesn't that require a proof? Yes it does. Here it is:

Proof of $A A^{-1}=I ; A^{-1} A=I . \quad$ Recall that, for $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$,

$$
\begin{aligned}
A^{-1} & =\frac{1}{a d-b c} \operatorname{adj} A, \quad \text { where } \\
\operatorname{adj} A & =\left[\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right]
\end{aligned}
$$

So it makes sense to first calculate $A(\operatorname{adj} A)$ and $(\operatorname{adj} A) A$ each:

$$
\begin{aligned}
A(\operatorname{adj} A) & =\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]\left[\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right] \\
& =\left[\begin{array}{ll}
a d+b(-c) & a(-b)+b a \\
c d+d(-c) & c(-b)+d a
\end{array}\right] \\
& =\left[\begin{array}{cc}
a d-b c & 0 \\
0 & a d-b c
\end{array}\right] \\
& =(a d-b c)\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]=(a d-b c) I
\end{aligned}
$$

and

$$
\begin{aligned}
(\operatorname{adj} A) A & =\left[\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right]\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \\
& =\left[\begin{array}{cc}
d a+(-b) c & d b+(-b) d \\
(-c) a+a c & (-c) b+a d
\end{array}\right] \\
& =\left[\begin{array}{cc}
a d-b c & 0 \\
0 & a d-b c
\end{array}\right] \\
& =(a d-b c)\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]=(a d-b c) I
\end{aligned}
$$

In short,

$$
A(\operatorname{adj} A)=(a d-b c) I, \quad \text { and } \quad(\operatorname{adj} A) A=(a d-b c) I
$$

Now, suppose $a d-b c \neq 0$. Then you can divide the two sides of each of the above two equalities:

$$
A \underbrace{A\left(\frac{1}{a d-b c} \operatorname{adj} A\right)}_{\|}=I, \quad \text { and } \quad \underbrace{\left(\frac{1}{a d-b c} \operatorname{adj} A\right)}_{\|} A=I \text {. }
$$

So, we indeed arrive at

$$
A A^{-1}=I, \quad \text { and } \quad A^{-1} A=I
$$

Let me highlight the result:

Fact 2. For $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right], \quad$ suppose

$$
\operatorname{det} A \neq 0, \quad \text { namely }, \quad a d-b c \neq 0
$$

Then

$$
A A^{-1}=I, \quad \text { and } \quad A^{-1} A=I
$$

- All right. What does this entail? Yes:

```
"We always have
\((* *) \quad a a^{-1}=1, \quad\) and \(\quad a^{-1} a=1\)
\(\xlongequal{\text { for any number } a \text {, provided } a \neq 0 \text {. (Right?) In the same token, }}\)
(\#\#) \(\quad A A^{-1}=I, \quad\) and \(\quad A^{-1} A=I\)
    \(\xlongequal{\text { for any matrix } A, \text { provided } \operatorname{det} A \neq 0 \text {. These two, (**) and }}\)
    \(\xlongequal{\text { (\#\#), are entirely parallel." }}\)
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- Now, so far things are relatively low key. But as is always true in math, things can suddenly take a dramatic turn. If you have a good mathematical insight, you already should have "the next level question" in mind, which is as follows:
- The next level question. Consider two matrices

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \quad \text { and } \quad B=\left[\begin{array}{cc}
p & q \\
r & s
\end{array}\right]
$$

Suppose their product $A B$ equals $I: A B=I$. Then is it true

$$
\begin{gather*}
B A=I ?  \tag{1}\\
B=A^{-1}, \quad \text { and } \quad A=B^{-1} ? \tag{2}
\end{gather*}
$$

- This question might at first sound a minor tweak of the last questions (whose answers are summarized in 'Fact 1 ' on page 4 , and 'Fact 2' on page 6). But the truth is, this time around the level of difficulty has been raised by a few notches. As always, let me first give the answers, and then dissect. The answers are
- 'Yes' for part (1).
- 'Yes' for part (2).
- All right. But why is that the case? Also, what do I really mean by saying that this one tops 'Fact 1' (page 4) and 'Fact 2' (page 6), in terms of the level of difficulty? You might just echo the voices of Student A and Student B below:


## $\star$ Student A.

"We already know $A A^{-1}=I$. That means $A X=I$, regarded as an equation with $X$ unknown, is solved as $X=A^{-1}$. Likewise, we already know $B^{-1} B=I$. That means $Y B=I$, once again regarded as an equation with $Y$ unknown, is solved as $Y=B^{-1}$. That answers this question entirely. So, what's really in there?"

- Wow. That sure sounds a very intelligent answer. Now, Student B:


## * Student B.

"If two numbers $a$ and $b$ satisfy $a b=1$ then, needless to say, $a$ and $b$ are reciprocals of each other: $b=a^{-1}$ and $a=b^{-1}$. So, in the same token, if you replace $a$ and $b$ with matrices, the same must be true."

- That sounds an intelligent answer too. Here are my comments:
"You two are actually not too off-the-mark. Congrats, you both have earned some extra credit for taking the courage to share your thoughts on this issue. Now, I still have to critique you, as follows:"


## * My retort to Student A:

"You are half-right. True, $A A^{-1}=I$ ensures that $A X=I$ has 'at least one' solution, which is, $X=A^{-1}$. But how are you so sure there is no other solution for $A X=I$ ? Another point to quibble: Your logic indeed works as long as $A^{-1}$ exists. How do you ensure that $A^{-1}$ exists, from the given condition $A B=I$ ? Justification for that is lacking."

## $\star$ My retort to Student B:

"You too are basically right, but you still need to justify that you can extrapolate your argument for matrices without a glitch. When you generalize something that holds true for numbers to something other than numbers (such as matrices), the odds are that you are going to lose some of the properties that the numbers possessed. For example, I have already stressed that, in general, $A B$ and $B A$ are not equal for matrices $A$ and $B$. So, nothing is really certain."

Wow, that was harsh. But like I said, the two students are still not too off-themark. Indeed, as Student B suggests, we must shoot for an extrapolation argument. Follow the next paragraph carefully:

## Outline of the extrapolation argument.

"First, $a b=1$ forces $a$ to be non-zero. Thus $a^{-1}$ exists. Then multiply $a^{-1}$ to the two sides of $a b=1$ : Then you immediately obtain $b=a^{-1}$. Similarly, by multiplying $b^{-1}$ instead of $a^{-1}$ you will obtain $a=b^{-1}$. The very same logic can be employed for matrices to pull the same conclusion for matrices, save that there are a couple of points which prove to be subtle ( $\# 1$ and $\# 2$ below)."

Point of subtlety \# 1: The extrapolation of the part

- "ab=1 forces a to be non-zero".

The right extrapolation of this statement for matrices is

$$
\text { - " } A B=I \text { forces } A \text { to have a non-zero determinant." }
$$

This latter statement is true. However, it is not that trivial. We need to provide a proof of it. (Here we go again!) For that matter, we in turn need to rely on a so-called "Product Formula". So, naturally, "Product Formula" is the next agenda on our check-list.

Point of subtlety \# 2: The extrapolation of the part

$$
\text { - "multiply } a^{-1} \text { to the two sides of } a b=1 \text { to get } b=a^{-1} \text {." }
$$

The right extrapolation of this statement for matrices is

- "multiply $A^{-1}$ to the two sides of $A B=I$ to get $B=A^{-1}$."

A couple of delicate points here: First you need to say you multiply $A^{-1} \xlongequal{\text { from the left, }}$ as in

$$
A^{-1}(A B)=A^{-1} I
$$

Second, you want to say $A^{-1}(A B)$ is reduced to $B$. However, technically speaking, in order to be able to safely claim that, you need to know in advance

$$
A^{-1}(A B)=\left(A^{-1} A\right) B
$$

This turns out to be true, indeed, more generally,

$$
A(B C)=(A B) C
$$

holds true for three matrices $A, B$ and $C$. Now, this last cited fact is something that requires a proof. (Here we go again!) This property $A(B C)=(A B) C$ is called the "Associativity Law". That's in our next agenda too.

- So, in sum, we need to cover
- Product Formula, and ○ Associativity Law.

These are coming up.

