Math 290 ELEMENTARY LINEAR ALGEBRA

REVIEW OF LECTURES – XXVIII

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• Rotations, reflections.

Today I want to start with one specific type of a matrix. What do the following matrices have in common?

$$\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & \frac{-\sqrt{2-\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{\sqrt{2+\sqrt{2}}}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

• As for the first five, check this out:

$$\cos \frac{\pi}{3} = \frac{1}{2}, \qquad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \\ \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \\ \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \qquad \sin \frac{\pi}{6} = \frac{1}{2}, \\ \cos \frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}, \qquad \sin \frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4}, \\ \cos \frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}, \qquad \sin \frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

$$\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix},$$
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix},$$
$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix},$$
$$\begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix} = \begin{bmatrix} \cos\frac{2\pi}{5} & -\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{bmatrix},$$
$$\begin{bmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & \frac{-\sqrt{2-\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{\sqrt{2+\sqrt{2}}}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{8} & -\sin\frac{\pi}{8} \\ \sin\frac{\pi}{8} & \cos\frac{\pi}{8} \end{bmatrix}.$$

• As for the last three,

$$\cos 0 = 1,$$
 $\sin 0 = 0,$
 $\cos \frac{\pi}{2} = 0,$
 $\sin \frac{\pi}{2} = 1,$
 $\cos \pi = -1,$
 $\sin \pi = 0.$

 So

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{bmatrix},$$
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{bmatrix}.$$

So

In short, the matrices listed on page 1 all fall into

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Let's calculate the characteristic polynomial, and the eigenvalues.

$$\chi_A(\lambda) = \det(\lambda I - A)$$

$$= \begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix}$$

$$= (\lambda - \cos\theta)(\lambda - \cos\theta) - (\sin\theta)(-\sin\theta)$$

$$= (\lambda - \cos\theta)^2 + (\sin\theta)^2$$

$$= \lambda^2 - 2(\cos\theta)\lambda + (\cos\theta)^2 + (\sin\theta)^2.$$

Here, recall

Formula.
$$\left(\cos\theta\right)^2 + \left(\sin\theta\right)^2 = 1.$$

We utilize this and conclude

$$\chi_A(\lambda) = \lambda^2 - 2(\cos\theta)\lambda + 1.$$

Summary. $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ has the following characteristic polynomial:

$$\chi_A(\lambda) = \lambda^2 - 2(\cos\theta)\lambda + 1.$$

• Can you find the eigenvalue(s) of A? Sure. What should we do? Yes, rely on

Quadratic formula. The equation

$$a x^2 + b x + c = 0$$
 $\left(a \neq 0\right)$

 $(x: \underline{\text{unknown}}, a, b, c: \underline{\text{knowns}})$ are solved as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The equation to solve is

$$\lambda^2 - 2\Big(\cos\theta\Big)\lambda + 1 = 0.$$

So, a = 1, $b = -2(\cos \theta)$, c = 1. Accordingly:

$$\lambda = \frac{-\left(-2\left(\cos\theta\right)\right) \pm \sqrt{\left(-2\left(\cos\theta\right)\right)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{2\left(\cos\theta\right) \pm \sqrt{4\left(\cos\theta\right)^2 - 4}}{2}$$

$$= \frac{2\left(\cos\theta\right) \pm 2\sqrt{\left(\cos\theta\right)^2 - 1}}{2}$$

$$= \frac{2\left(\cos\theta\right) \pm 2\sqrt{-\left(\sin\theta\right)^2}}{2} \qquad \text{(by Formula)}$$

$$= \left(\cos\theta\right) \pm \sqrt{-\left(\sin\theta\right)^2}.$$

$$4$$

So, how should we handle $\pm \sqrt{-(\sin \theta)^2}$? Can we simplify it? Yes, there is a dichotomy:

(i) If
$$\sin \theta$$
 is 0, then $\pm \sqrt{-\left(\sin \theta\right)^2} = 0$

(ii) If
$$\sin \theta$$
 is non-zero, then $(\sin \theta)^2$ is positive, thus $-(\sin \theta)^2$ is negative. Then $\pm \sqrt{-(\sin \theta)^2}$ is 'non-real'. Write it as $\pm \sqrt{-1} \sin \theta$.

Here, suppose $\sin \theta = 0$ (namely, suppose we are in case (i)). Then $\pm \sqrt{-(\sin \theta)^2}$ still equals $\pm \sqrt{-1} \sin \theta$, indeed, both equal 0. So in both cases (i) and (ii) we end up getting

$$\pm \sqrt{-\left(\sin\theta\right)^2} = \pm \sqrt{-1}\sin\theta.$$

Summary 2. $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ has the following eigenvalues: $\lambda = (\cos \theta) \pm \sqrt{-1} (\sin \theta).$

• Next, let's find the eigenvectors of A, associated with the above eigenvalues. In what follows, the double sign in the same order.

Since
$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$
, the equation $A\boldsymbol{x} = \left(\left(\cos \theta \right) \pm \sqrt{-1} \left(\sin \theta \right) \right) \boldsymbol{x}$ is

(#)
$$\begin{bmatrix} \cos\theta & -\sin\theta\\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x\\ y \end{bmatrix} = \left(\left(\cos\theta \right) \pm \sqrt{-1} \left(\sin\theta \right) \right) \begin{bmatrix} x\\ y \end{bmatrix}.$$

That is,

$$\begin{cases} \left(\cos\theta\right)x - \left(\sin\theta\right)y = \left(\left(\cos\theta\right) \pm \sqrt{-1}\left(\sin\theta\right)\right)x, \\ \left(\sin\theta\right)x + \left(\cos\theta\right)y = \left(\left(\cos\theta\right) \pm \sqrt{-1}\left(\sin\theta\right)\right)y. \end{cases}$$

Shift the terms:

$$\begin{cases} \mp \sqrt{-1} \left(\sin \theta \right) x & - \left(\sin \theta \right) y = 0, \\ \left(\sin \theta \right) x \mp \sqrt{-1} \left(\cos \theta \right) y = 0. \end{cases}$$

These equations are the essentially identical. You may or may not see it immediately. But actually the second equation is obtained by just multiplying $\pm \sqrt{-1}$ to the two sides of the first equation. Indeed:

$$\begin{pmatrix} \pm \sqrt{-1} \end{pmatrix} \left(\mp \sqrt{-1} \left(\sin \theta \right) x - \left(\sin \theta \right) y \right)$$

$$= \left(\pm \sqrt{-1} \right) \left(\mp \sqrt{-1} \right) \left(\sin \theta \right) x - \left(\pm \sqrt{-1} \right) \left(\sin \theta \right) y$$

$$= -\left(\sqrt{-1} \right)^2 \left(\sin \theta \right) x \mp \sqrt{-1} \left(\sin \theta \right) y$$

$$= -\left(-1 \right) \left(\sin \theta \right) x \mp \sqrt{-1} \left(\sin \theta \right) y$$

$$= \left(\sin \theta \right) x \mp \sqrt{-1} \left(\sin \theta \right) y.$$

So, ignore the second equation:

$$\mp \sqrt{-1} \left(\sin \theta \right) x - \left(\sin \theta \right) y = 0.$$

Assume $\sin \theta \neq 0$, and divide the two sides by $\sin \theta$:

$$\mp \sqrt{-1} x - y = 0.$$

Clearly $x = \sqrt{-1}$, $y = \pm 1$ works. Thus:

• $\boldsymbol{x}_{\pm} = \begin{bmatrix} \sqrt{-1} \\ \pm 1 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue

$$\lambda = \left(\cos\theta\right) \pm \sqrt{-1} \left(\sin\theta\right).$$

Diagonalization result. $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ is diagonalized as follows:

$$Q^{-1}AQ = \begin{bmatrix} \left(\cos\theta\right) + \sqrt{-1}\left(\sin\theta\right) & 0\\ 0 & \left(\cos\theta\right) - \sqrt{-1}\left(\sin\theta\right) \end{bmatrix},$$

where $Q = \begin{bmatrix} \sqrt{-1} & \sqrt{-1}\\ 1 & -1 \end{bmatrix}.$

Example 1. We can diagonalize

$$A = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix},$$

as follows: As we have already observed:

$$\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}.$$

So the eigenvalues of A are

$$\left(\cos\frac{\pi}{3}\right) \pm \sqrt{-1} \left(\sin\frac{\pi}{3}\right) = \frac{1}{2} \pm \sqrt{-1} \frac{\sqrt{3}}{2}$$
$$= \frac{1 \pm \sqrt{-3}}{2}.$$

Accordingly:

$$Q^{-1}AQ = \begin{bmatrix} \frac{1+\sqrt{-3}}{2} & 0\\ 0 & \frac{1-\sqrt{-3}}{2} \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} \sqrt{-1} & \sqrt{-1}\\ 1 & -1 \end{bmatrix}.$$

Example 2. We can diagonalize

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

as follows: As we have already observed:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}.$$

So the eigenvalues of A are

$$\left(\cos\frac{\pi}{2}\right) \pm \sqrt{-1} \left(\sin\frac{\pi}{2}\right) = 0 \pm \sqrt{-1} \cdot 1$$
$$= \pm \sqrt{-1}.$$

Accordingly

$$Q^{-1}AQ = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}$$
, where $Q = \begin{bmatrix} \sqrt{-1} & \sqrt{-1} \\ 1 & -1 \end{bmatrix}$.

Exercise 1. Diagonalize each of

(1)
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$
, (2) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$,

(3)
$$\begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}, \quad (4)$$

$$\begin{bmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & \frac{-\sqrt{2-\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{\sqrt{2+\sqrt{2}}}{2} \end{bmatrix}$$

•

Example 3. Let's take a look at

$$A = \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

Actually this A still falls into

$$\begin{bmatrix} \cos\theta & -\sin\theta\\ \sin\theta & \cos\theta \end{bmatrix}.$$

The underlying reason behind it is,

$$3^2 + 4^2 = 5^2$$
.

 So

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1.$$

In calculus, we have learned the following fundamental fact:

Fact. Let *a* and *b* be real numbers. Suppose they satisfy

$$a^2 + b^2 = 1.$$

Then there exists a real number θ such that

$$a = \cos \theta, \qquad b = \sin \theta.$$

The above falls precisely into the case $a = \frac{3}{5}$ and $b = \frac{4}{5}$. The problem is, for these a, b, you cannot express θ concretely as a concrete number times π . Indeed, θ has at least three expressions:

$$\theta = \arccos \frac{3}{5} = \arcsin \frac{4}{5} = \arctan \frac{4}{3}.$$

However, none of these three can be simplified any further. The good news is, nevertheless, we can find the eigenvalues of A, and also diagonalize A, following the above method. Let's perform: The eigenvalues of $A = \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ are $\left(\cos\theta\right) \pm \sqrt{-1}\left(\sin\theta\right) = \frac{3}{5} \pm \sqrt{-1} \cdot \frac{4}{5}$ $= \frac{3 \pm \sqrt{-1} \cdot 4}{5}.$

We may diagonalize A as

$$Q^{-1}AQ = \begin{bmatrix} \frac{3+\sqrt{-1}\cdot 4}{5} & 0\\ 0 & \frac{3-\sqrt{-1}\cdot 4}{5} \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} \sqrt{-1}\sqrt{-1}\\ 1 & -1 \end{bmatrix}.$$

Remark. In the above, we've seen

$$3^2 + 4^2 = 5^2.$$

There is a way to produce infinitely many triplets (a, b, c) where a, b and c are all integers, satisfying

$$a^2 + b^2 = c^2.$$

Indeed, just substitute integers k into

$$\begin{cases} a = k^2 - 1, \\ b = 2k, \\ c = k^2 + 1. \end{cases}$$

Substitute k = 2 into the above and you'll end up getting a = 3, b = 4, c = 5. Exercise 2. Substitute k = 3, 4, 5, 6 each, into the above to produce triplets

$$(a, b, c)$$
 satisfying $a^2 + b^2 = c^2$. Then write out the matrix $\begin{bmatrix} \frac{a}{c} & \frac{-b}{c} \\ \frac{b}{c} & \frac{a}{c} \end{bmatrix}$ in each of the four cases. Find the eigenvalues. Diagonalize.

• Kissing cousins.

Here is today's second theme:

$$B = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}.$$

What about it? We have just dealt with it, or what? On a second look, this is actually different from what we have just seen:

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Yes, B is different from A. Even if you change θ suitably, you cannot convert A into B. So, in other words, A and B are different animals, though they alwfully resemble each other. Since we have alanyzed A, let's also analyze B. So, eigenvalues and diagonalizations of B. Here we go:

$$\begin{split} \chi_B \left(\lambda \right) &= \det \left(\lambda I - B \right) \\ &= \begin{vmatrix} \lambda - \cos \theta & -\sin \theta \\ -\sin \theta & \lambda + \cos \theta \end{vmatrix} \\ &= \left(\lambda - \cos \theta \right) \left(\lambda + \cos \theta \right) - \left(-\sin \theta \right) \left(-\sin \theta \right) \\ &= \lambda^2 - \left(\left(\cos \theta \right)^2 + \left(\sin \theta \right)^2 \right) \\ &= \lambda^2 - 1 \\ &= \left(\lambda - 1 \right) \left(\lambda + 1 \right). \end{split}$$

In the above, we have used

a.
$$\left(\cos\theta\right)^2 + \left(\sin\theta\right)^2 = 1.$$

Formula.

Summary. $B = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$ has the following characteristic polynomial: $\chi_B(\lambda) = (\lambda - 1)(\lambda + 1).$

B has the following eigenvalues:

$$\lambda = \pm 1.$$

• Next, let's find the eigenvectors of B, associated with the above eigenvalues. In what follows, the double sign in the same order.

Since
$$B = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$$
, the equation $B\boldsymbol{x} = \pm \boldsymbol{x}$ is
(#) $\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \pm \begin{bmatrix} x \\ y \end{bmatrix}$.

That is,

$$\begin{pmatrix} \left(\cos\theta\right)x + \left(\sin\theta\right)y = \pm x, \\ \left(\sin\theta\right)x - \left(\cos\theta\right)y = \pm y.$$

Shift the terms:

$$\begin{cases} \left(\left(\cos\theta\right)\mp1\right)x + \left(\sin\theta\right)y = 0,\\ \left(\sin\theta\right)x + \left(\left(-\cos\theta\right)\mp1\right)y = 0. \end{cases}$$

These equations are the essentially identical. You may or may not see it immediately.

But actually the second equation is obtained by just multiplying $\frac{\left(-\cos\theta\right) \mp 1}{\sin\theta}$ to the two sides of the first equation (where $\sin\theta \neq 0$ is assumed). Indeed:

$$\frac{\left(-\cos\theta\right)\mp 1}{\sin\theta} \left[\left(\left(\cos\theta\right)\mp 1\right)x + \left(\sin\theta\right)y \right]$$

$$= \frac{1}{\sin\theta} \cdot \left[\left(\left(-\cos\theta\right)\mp 1\right)\left(\left(\cos\theta\right)\mp 1\right)x + \left(\left(-\cos\theta\right)\mp 1\right)\left(\sin\theta\right)y \right]$$

$$= \frac{1}{\sin\theta} \cdot \left[\left(-\left(\cos\theta\right)^2 + \left(\mp 1\right)^2\right)x + \left(\left(-\cos\theta\right)\mp 1\right)\left(\sin\theta\right)y \right]$$

$$= \frac{1}{\sin\theta} \cdot \left[\left(1 - \left(\cos\theta\right)^2\right)x + \left(\left(-\cos\theta\right)\mp 1\right)\left(\sin\theta\right)y \right]$$

$$= \frac{1}{\sin\theta} \cdot \left[\left(\sin\theta\right)^2x + \left(\left(-\cos\theta\right)\mp 1\right)\left(\sin\theta\right)y \right]$$

$$= \left(\sin\theta\right)x + \left(\left(-\cos\theta\right)\mp 1\right)y.$$

So, ignore the second equation:

$$\left(\left(\cos\theta\right)\mp1\right)x+\left(\sin\theta\right)y = 0.$$

Clearly $x = \sin \theta$, $y = -(\cos \theta) \pm 1$ works. Thus:

 $\circ \ \boldsymbol{x}_{\pm} = \begin{bmatrix} \sin \theta \\ -(\cos \theta) \pm 1 \end{bmatrix} \text{ is an eigenvector of } B \text{ associated with} \\ \lambda = \pm 1.$

Diagonalization result. $B = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$ is diagonalized as follows: $Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, where $Q = \begin{bmatrix} \sin \theta & \sin \theta \\ -(\cos \theta) + 1 & -(\cos \theta) - 1 \end{bmatrix}$.

• Now agree that all of the following fall into the *B*-type:

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{1-\sqrt{5}}{4} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & \frac{\sqrt{2-\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{-\sqrt{2+\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{-\sqrt{2+\sqrt{2}}}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Exercise 3. Diagonalize each of the above eight matrices.

Exercise 4. Do the same as Exercise 2, but with $\begin{bmatrix} \frac{a}{c} & \frac{b}{c} \\ \frac{b}{c} & \frac{-a}{c} \end{bmatrix}$

instead.

Definition (Orthogonal matrices).

Matrices of the forms

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix},$$

(θ : a real number) are called orthogonal matrices.

• We are going to explore the properties of orthogonal matrices in the next lecture. Also, I plan to introduce symmetric matrices, and explain how they and orthogonal matrices have bearings of each other in the context of diagonalizability.

- Answers to Exercise problems.
- $\circ~$ Exercise 1 answers (page 8).

(1)
$$Q^{-1}AQ = \begin{bmatrix} \frac{1+\sqrt{-1}}{\sqrt{2}} & 0\\ 0 & \frac{1+\sqrt{-1}}{\sqrt{2}} \end{bmatrix}$$
, where $Q = \begin{bmatrix} \sqrt{-1} & \sqrt{-1}\\ 1 & -1 \end{bmatrix}$.

(2)
$$Q^{-1}AQ = \begin{bmatrix} \frac{\sqrt{3} + \sqrt{-1}}{2} & 0\\ 0 & \frac{\sqrt{3} - \sqrt{-1}}{2} \end{bmatrix}$$
, where $Q = \begin{bmatrix} \sqrt{-1} & \sqrt{-1}\\ 1 & -1 \end{bmatrix}$.

(3)
$$Q^{-1}AQ = \begin{bmatrix} \frac{-1+\sqrt{5}+\sqrt{-10-2\sqrt{5}}}{4} & 0\\ 0 & \frac{-1+\sqrt{5}-\sqrt{-10-2\sqrt{5}}}{4} \end{bmatrix}$$

where
$$Q = \begin{bmatrix} \sqrt{-1} & \sqrt{-1} \\ 1 & -1 \end{bmatrix}$$
.

(4)
$$Q^{-1}AQ = \begin{bmatrix} \frac{\sqrt{2+\sqrt{2}} + \sqrt{-2+\sqrt{2}}}{2} & 0\\ 0 & \frac{\sqrt{2+\sqrt{2}} - \sqrt{-2+\sqrt{2}}}{2} \end{bmatrix},$$

where $Q = \begin{bmatrix} \sqrt{-1} & \sqrt{-1} \\ 1 & -1 \end{bmatrix}$.

 $\circ~$ Exercise 2 answers (page 10).

$$\circ \ k = 3 \implies \left(a, \ b, \ c\right) = \left(8, \ 6, \ 10\right). \ A = \begin{bmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}. \ \lambda = \frac{4 \pm \sqrt{-1} \cdot 3}{5}.$$
$$Q^{-1}AQ = \begin{bmatrix} \frac{4 + \sqrt{-1} \cdot 3}{5} & 0 \\ 0 & \frac{4 - \sqrt{-1} \cdot 3}{5} \end{bmatrix}, \ \text{where} \ Q = \begin{bmatrix} \sqrt{-1} & \sqrt{-1} \\ 1 & -1 \end{bmatrix}.$$

$$\circ \quad k = 4 \implies \left(a, b, c\right) = \left(15, 8, 17\right). \quad A = \begin{bmatrix} \frac{15}{17} & \frac{-8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{bmatrix}. \quad \lambda = \frac{15 \pm \sqrt{-1} \cdot 8}{17}.$$

$$Q^{-1}AQ = \begin{bmatrix} \frac{15 + \sqrt{-1 \cdot 8}}{17} & 0\\ 0 & \frac{15 - \sqrt{-1 \cdot 8}}{17} \end{bmatrix}, \text{ where } Q = \begin{bmatrix} \sqrt{-1} \sqrt{-1}\\ 1 & -1 \end{bmatrix}.$$

•
$$k = 5 \implies (a, b, c) = (24, 10, 26).$$
 $A = \begin{bmatrix} \frac{12}{13} & \frac{-5}{13}\\ \frac{5}{13} & \frac{12}{13} \end{bmatrix}.$ $\lambda = \frac{12 \pm \sqrt{-1} \cdot 5}{13}.$

$$Q^{-1}AQ = \begin{bmatrix} \frac{12 + \sqrt{-1} \cdot 5}{13} & 0\\ 0 & \frac{12 - \sqrt{-1} \cdot 5}{13} \end{bmatrix}, \text{ where } Q = \begin{bmatrix} \sqrt{-1} \sqrt{-1}\\ 1 & -1 \end{bmatrix}.$$

$$\circ \quad k = 6 \implies \left(a, b, c\right) = \left(35, 12, 37\right). \quad A = \begin{bmatrix} \frac{35}{37} & \frac{-12}{37} \\ \frac{12}{37} & \frac{35}{37} \end{bmatrix}. \quad \lambda = \frac{35 \pm \sqrt{-1} \cdot 12}{37}.$$

$$Q^{-1}AQ = \begin{bmatrix} \frac{35 + \sqrt{-1} \cdot 12}{37} & 0\\ 0 & \frac{35 - \sqrt{-1} \cdot 12}{37} \end{bmatrix}, \text{ where } Q = \begin{bmatrix} \sqrt{-1} & \sqrt{-1}\\ 1 & -1 \end{bmatrix}.$$

• Exercise 3 answers (page 14).

 $\circ \qquad B = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix} \qquad \Longrightarrow \qquad$ $Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, where $Q = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ & \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$. $\circ \qquad B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \qquad \Longrightarrow \qquad$ $Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, where $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} + 1 & -\frac{1}{\sqrt{2}} - 1 \end{bmatrix}$. $\circ \qquad B = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix} \implies$ $Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, where $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{3}}{2} + 1 & -\frac{\sqrt{3}}{2} - 1 \end{bmatrix}$. $\circ \qquad B = \begin{vmatrix} \frac{-1+\sqrt{5}}{4} & \frac{\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{1-\sqrt{5}}{4} \end{vmatrix} \qquad \Longrightarrow \qquad$ $Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, where $Q = \begin{bmatrix} \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{\sqrt{10+2\sqrt{5}}}{4} \\ & & \\ 5-\sqrt{5} & -3-\sqrt{5} \end{bmatrix}$.

$$\circ \qquad B = \begin{bmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & \frac{\sqrt{2-\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{-\sqrt{2+\sqrt{2}}}{2} \end{bmatrix} \implies \\ Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{\sqrt{2-\sqrt{2}}}{2} \\ \frac{2-\sqrt{2+\sqrt{2}}}{2} & \frac{-2-\sqrt{2+\sqrt{2}}}{2} \end{bmatrix}.$$

$$\circ \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \implies Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\circ \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$\circ \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \implies Q^{-1}BQ = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

 $\circ~$ Exercise 4 answers (page 14).

$$\circ \qquad k = 3 \implies (a, b, c) = (8, 6, 10). \qquad B = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{-4}{5} \end{bmatrix}. \qquad \lambda = \pm 1.$$
$$Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{-9}{5} \end{bmatrix}.$$

$$\circ \qquad k = 4 \implies (a, b, c) = (15, 8, 17). \qquad B = \begin{bmatrix} \frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{-15}{17} \end{bmatrix}. \qquad \lambda = \pm 1.$$
$$Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} \frac{8}{17} & \frac{8}{17} \\ \frac{2}{17} & \frac{-32}{17} \end{bmatrix}.$$

$$\circ \qquad k = 5 \implies (a, b, c) = (24, 10, 26). \qquad B = \begin{bmatrix} \frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{-12}{13} \end{bmatrix}. \qquad \lambda = \pm 1.$$
$$Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} \frac{5}{13} & \frac{5}{13} \\ \frac{1}{13} & \frac{-25}{13} \end{bmatrix}.$$

$$\circ \qquad k = 6 \implies (a, b, c) = (35, 12, 37). \qquad B = \begin{bmatrix} \frac{35}{37} & \frac{12}{37} \\ \frac{12}{37} & \frac{-35}{37} \end{bmatrix}. \qquad \lambda = \pm 1.$$
$$Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} \frac{12}{37} & \frac{12}{37} \\ \frac{2}{37} & \frac{-72}{37} \end{bmatrix}.$$