# Math 290 ELEMENTARY LINEAR ALGEBRA REVIEW OF LECTURES - I 

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## §1. What is linear algebra? Overview.

Welcome to Math 290, 'Elementary Linear Algebra'. Right off the bat you see the adjective 'elementary' in the course title. Think of it as 'introduction to'. Indeed, in our course listings there are sequels to Math 290 - Math 590, and Math 790, that is. It's not that you are required to take them. Those are for math majors (an upper-division undergraduate course), and for Master and Ph.D. degree-seekers in math (a graduate course), respectively. Case in point: I know Math 290 class is usually populated with non-math majors/non-aspiring mathematicians. Here I am not neglecting anyone in this room: If you happen to be a math major, that's a blessing. So bear with me if you feel what I say next don't apply to you. In what follows I refer to the 'presumed majority' of this class simply as 'you' (in second person plural). From my years of experience I can safely bet you are not completely familiar with the nature of math. Sorry I'm sounding blunt, but this is not a criticism as you will know it if you care enough to listen. Here is what I mean: You see math as a mere pathway to whatever subject you will end up majoring in (engineering, bio-science, econ, ..), but math itself doesn't quite strike you as a 'real-deal', a discipline that has a robust, cutting-edge scientific content. And such an impression is enhanced by what you see on Day 1: Math 290 class typically starts (as it should) with "systems of linear equations". An example:

$$
\left\{\begin{array}{r}
4 x+3 y=5  \tag{*}\\
2 x-6 y=-7
\end{array}\right.
$$

Everyone understands what this signifies. It is a pair of equations, or a system of equations. Out of the blue, the first tip of the day: In linear algebra, for some (so far undisclosed) reason, we are inclined to bother to rewrite $(*)$ as

$$
\left[\begin{array}{cc}
4 & 3 \\
2 & -6
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{c}
5 \\
-7
\end{array}\right]
$$

So, once again:
$\left.(*) \quad\left\{\begin{array}{l}4 x+3 y=5, \\ 2 x-6 y=-7\end{array}\right\} \Longleftrightarrow \begin{array}{cc}4 & 3 \\ 2 & -6\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}5 \\ -7\end{array}\right]$

The box on the right involves some 'arrays' enclosed by the brackets [ ]. The array $\left[\begin{array}{cc}4 & 3 \\ 2 & -6\end{array}\right]$ is called a matrix . A $2 \times 2$ matrix, to be exact. 'Matrix' is a technical term:

$$
\text { matrix; } \quad \text { matrices }(p l)
$$

Also, the arrays $\left[\begin{array}{l}x \\ y\end{array}\right]$ and $\left[\begin{array}{c}5 \\ -7\end{array}\right]$ are called column vectors . Actually, column vectors are also considered as matrices. $2 \times 1$ matrices, to be exact. As I go like that, I know you have a lot of follow-up questions. But let's not ask questions right now, because the subsequent lectures will answer all of them. So today let's just stay low-key, all right? I somehow wanted to just throw the matrix equation, but without detailed explanation. Just know that this way of rewriting is quintessentially important in linear algebra. You are going to see a ton of these as the semester progresses. More on this later.

So, let's forget about matrices. Back to

$$
\left\{\begin{array}{l}
4 x+3 y=5  \tag{*}\\
2 x-6 y=-7
\end{array}\right.
$$

You are supposed to solve it for $x$ and $y$. With a pencil and a scratch paper, you can pull the answer easily with a relatively short amount of time:

$$
(x, y)=\left(\frac{3}{10}, \frac{19}{15}\right)
$$

Oh, by the way, I know some of you will insist that something like this should better be handled by a calculator, it makes little sense we humans impose on ourselves to do the job. Guess what, that is actually one of the salient points. This is one of the

FAQs by students. Um, to those who are strong advocators of the notion that math is becoming obsolete thanks to the computers, I plainly and boldly refute that, with an imperishable logic. I'm more than eager to engage in that debate, but not now. Later. Hold on to that tought.

Now, that temporarily aside, the above is some high school stuff. Right? If you get that impression, I don't blame you. However, if you think that not much else is what Math 290 is all about, so this class is not officially a remedial course (= developmental course) per se, but it is in reality something akin to it. You say that would be so underwhelming. You are here only because of the university requirement for your degree, but you feel this class is redundant/useless because there is not much to learn. Because you are some knowledgeable smartass.

Listen, folks. Such a claim (except the last part) is barely tenable (as in I'm going to bail you out on this one) and that's only because you have never eyewitnessed how easily simple math problems can be tweaked into something of a putative theoretical science, where I use the term 'science' to mean 'human endeavors to find answers to open questions' - from riddles to conundrums to world famous unanswerable paradoxes. The fact of the matter is there indeed are a myriad of unanswered questions in math, whether you believe me or not. And I didn't make this one up. Professional research mathematicians (like myself and my colleagues in KU Math Department) are literally devoting their lives to tackling those.

But today I don't want to go that far. Today I just want to demonstrate how the aforementioned rudimentary math problem can 'evolve'. Today is Day 1 so, in a day like this, I can afford to spend time to go out of my way to prove that this class is worth a semester of your time. How does that sound?

First phase of generalization. The first phase of generalization of (*) would be to make it into

$$
\left\{\begin{array}{l}
a x+b y=e \\
c x+d y=f
\end{array}\right.
$$

So, the numbers are replaced by the letters. This can be solved for $x$ and $y$. I won't show you the steps, because like I said, today is just a sneak-preview, so let's cut to the chase and jump to the answer:

$$
(x, y)=\left(\frac{-b f+d e}{a d-b c}, \frac{a f-c e}{a d-b c}\right)
$$

Notice that the answer involves fractions. The denominators of the two fractions are both $a d-b c$. This answer makes sense provided $a d-b c$ does not equal 0 :

$$
a d-b c \neq 0
$$

So, in other words, technically, we still need to address the problem when

$$
a d-b c=0
$$

happens. Today let's not go to that direction.

Formula. The system of equations
(\#)

$$
\left\{\begin{array}{l}
a x+b y=e \\
c x+d y=f
\end{array}\right.
$$

is solved as

$$
(x, y)=\left(\frac{-b f+d e}{a d-b c}, \frac{a f-c e}{a d-b c}\right)
$$

provided

$$
a d-b c \neq 0 .
$$

Now you can effectively use this formula and solve any system of linear equations of the form (\#) where $a, b, c, d, e$ and $f$ are concrete numbers, provided $a, b$, $c$ and $d$ satisfy $a d-b c \neq 0$. Fair enough.

Exercise 1. Solve ( $*$ ) on page 2 using Formula above. For that matter, simply substitute

$$
a=4, \quad b=3, \quad c=2, \quad d=-6, \quad e=5 \quad \text { and } \quad f=-7
$$

into each of $\frac{-b f+d e}{a d-b c}$ and $\frac{a f-c e}{a d-b c}$, and simplify.

Second phase of generalization. Let's take a look at

$$
\left\{\begin{array}{r}
2 x_{1}-x_{2}+4 x_{3}=1  \tag{**}\\
x_{1}+2 x_{2}+5 x_{3}=2 \\
3 x_{1}-x_{2}+2 x_{3}=4
\end{array}\right.
$$

This one has three equations and three unknowns. Once again, without asking questions about the juxtaposing of the arrays or the meaning of two quantities that are made of arrays being equal, just accept the fact that in linear algebra we tend to rewrite ( $* *$ ) as

$$
\left[\begin{array}{ccc}
2 & -1 & 4 \\
1 & 2 & 5 \\
3 & -1 & 2
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{l}
1 \\
2 \\
4
\end{array}\right]
$$

Once again:

$$
\left\{\begin{array}{r}
2 x_{1}-x_{2}+4 x_{3}=1, \\
x_{1}+2 x_{2}+5 x_{3}=2, \\
3 x_{1}-x_{2}+2 x_{3}=4
\end{array}\right\} \stackrel{\text { "equivalent" }}{\Longleftrightarrow}\left[\begin{array}{ccc}
2 & -1 & 4 \\
1 & 2 & 5 \\
3 & -1 & 2
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{l}
1 \\
2 \\
4
\end{array}\right]
$$

You probably see this coming: The array $\left[\begin{array}{ccc}2 & -1 & 4 \\ 1 & 2 & 5 \\ 3 & -1 & 2\end{array}\right]$ is called a $(3 \times 3) \underline{\text { matrix }}$. Also, the arrays $\left[\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right]$ and $\left[\begin{array}{l}1 \\ 2 \\ 4\end{array}\right]$ are called column vectors (of length 3), or alternatively, $3 \times 1$ matrices.

Now, how to solve this system of equations is a different story altogether, though you might have practiced solving this kind of problems before somewhere. I know students are always so keen to know the part "how to solve". Students are smart enough (I mean it) to know that that's the most relevant thing when it comes to exams. (Know that only a part of my exams are about "how to solve". Profound understanding of the logical structure/interdependence of the lectures will always be paramount.) Now here is the thing. "How to solve" typically involves a formula. Naturally, students ask for a formula. Fair enough. So why stall? Here it is:

Formula. The system of equations

$$
\left\{\begin{aligned}
a_{1} x_{1}+a_{2} x_{2}+a_{3} x_{3} & =p \\
b_{1} x_{1}+b_{2} x_{2}+b_{3} x_{3} & =q \\
c_{1} x_{1}+c_{2} x_{2}+c_{3} x_{3} & =r
\end{aligned}\right.
$$

is solved as

$$
\begin{aligned}
\left(x_{1}, x_{2}, x_{3}\right)= & \frac{p b_{2} c_{3}-p b_{3} c_{2}-a_{2} q c_{3}+a_{2} b_{3} r+a_{3} q c_{2}-a_{3} b_{2} r}{a_{1} b_{2} c_{3}-a_{1} b_{3} c_{2}-a_{2} b_{1} c_{3}+a_{2} b_{3} c_{1}+a_{3} b_{1} c_{2}-a_{3} b_{2} c_{1}} \\
& \frac{a_{1} q c_{3}-a_{1} b_{3} r-p b_{1} c_{3}+p b_{3} c_{1}+a_{3} b_{1} r-a_{3} q c_{1}}{a_{1} b_{2} c_{3}-a_{1} b_{3} c_{2}-a_{2} b_{1} c_{3}+a_{2} b_{3} c_{1}+a_{3} b_{1} c_{2}-a_{3} b_{2} c_{1}} \\
& \left.\frac{a_{1} b_{2} r-a_{1} q c_{2}-a_{2} b_{1} r+a_{2} q c_{1}+p b_{1} c_{2}-p b_{2} c_{1}}{a_{1} b_{2} c_{3}-a_{1} b_{3} c_{2}-a_{2} b_{1} c_{3}+a_{2} b_{3} c_{1}+a_{3} b_{1} c_{2}-a_{3} b_{2} c_{1}}\right)
\end{aligned}
$$

$\underline{\underline{\text { provided }}}$

$$
a_{1} b_{2} c_{3}-a_{1} b_{3} c_{2}-a_{2} b_{1} c_{3}+a_{2} b_{3} c_{1}+a_{3} b_{1} c_{2}-a_{3} b_{2} c_{1} \neq 0
$$

Notice the three fractions in the answer have the same denominator. I stress that, therefore, this formula works, provided that denominator does not equal 0 .

Is this intimidating? A little? Yes I agree. But the truth is, this works. You don't have to memorize it today (unless you can't help it). But rather, I want you to just stare at it. You can detect some patterns in the formations of this formula. The second tip of the day: In linear algebra, there are certain basic patterns that you naturally run into all the time. Those patterns are so natural, and so ubiquitous, that you cannot really get away from them. A part of your job in this class, therefore, is to familiarize yourself with those patterns. Like I said, the above formula clearly manifests some patterns. There is actually a concept that best captures those patterns, called the 'determinant ':
determinant.

Remember that earlier we saw that the quantity $a d-b c$ naturally popped up out of

$$
\left\{\begin{array}{l}
a x+b y=e \\
c x+d y=f
\end{array}\right.
$$

Now the $3 \times 3$ counterpart of $a d-b c$ is

$$
a_{1} b_{2} c_{3}-a_{1} b_{3} c_{2}-a_{2} b_{1} c_{3}+a_{2} b_{3} c_{1}+a_{3} b_{1} c_{2}-a_{3} b_{2} c_{1}
$$

which pops out of

$$
\left\{\begin{aligned}
a_{1} x_{1}+a_{2} x_{2}+a_{3} x_{3} & =p \\
b_{1} x_{1}+b_{2} x_{2}+b_{3} x_{3} & =q \\
c_{1} x_{1}+c_{2} x_{2}+c_{3} x_{3} & =r
\end{aligned}\right.
$$

where the latter is the $3 \times 3$ counterpart of

$$
\left\{\begin{array}{l}
a x+b y=e \\
c x+d y=f
\end{array}\right.
$$

Today I don't want to pack the lecture with too much information, and I realize I'm getting a little ahead of myself. But since I'm paving the way for an exploration of the gist of basic linear algebra, and am already half the way through, why not go all the way and throw the following? Again, you don't have to memorize it today (unless you can't help it):

- $a d-b c$ is called the determinant of the $2 \times 2$ matrix $\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$.
- $\quad a_{1} b_{2} c_{3}-a_{1} b_{3} c_{2}-a_{2} b_{1} c_{3}+a_{2} b_{3} c_{1}+a_{3} b_{1} c_{2}-a_{3} b_{2} c_{1}$
is called the determinant of the $3 \times 3$ matrix $\left[\begin{array}{lll}a_{1} & a_{2} & a_{3} \\ b_{1} & b_{2} & b_{3} \\ c_{1} & c_{2} & c_{3}\end{array}\right]$.

Now, back to the formula on page 6. You can effectively use it and solve (**) on page 5:

$$
\left\{\begin{aligned}
2 x_{1}-x_{2}+4 x_{3} & =1 \\
x_{1}+2 x_{2}+5 x_{3} & =2 \\
3 x_{1}-x_{2}+2 x_{3} & =4
\end{aligned}\right.
$$

by way of just throwing

$$
\begin{array}{lll}
a_{1}=2, & a_{2}=-1, & a_{3}=4, \\
b_{1}=1, & b_{2}=2, & b_{3}=5, \\
c_{1}=3, & c_{2}=-1, & c_{3}=2, \\
p=1, & q=2 \text { and } r=4
\end{array}
$$

into

$$
\begin{aligned}
\left(x_{1}, x_{2}, x_{3}\right)=( & \frac{p b_{2} c_{3}-p b_{3} c_{2}-a_{2} q c_{3}+a_{2} b_{3} r+a_{3} q c_{2}-a_{3} b_{2} r}{a_{1} b_{2} c_{3}-a_{1} b_{3} c_{2}-a_{2} b_{1} c_{3}+a_{2} b_{3} c_{1}+a_{3} b_{1} c_{2}-a_{3} b_{2} c_{1}} \\
& \frac{a_{1} q c_{3}-a_{1} b_{3} r-p b_{1} c_{3}+p b_{3} c_{1}+a_{3} b_{1} r-a_{3} q c_{1}}{a_{1} b_{2} c_{3}-a_{1} b_{3} c_{2}-a_{2} b_{1} c_{3}+a_{2} b_{3} c_{1}+a_{3} b_{1} c_{2}-a_{3} b_{2} c_{1}}, \\
& \left.\frac{a_{1} b_{2} r-a_{1} q c_{2}-a_{2} b_{1} r+a_{2} q c_{1}+p b_{1} c_{2}-p b_{2} c_{1}}{a_{1} b_{2} c_{3}-a_{1} b_{3} c_{2}-a_{2} b_{1} c_{3}+a_{2} b_{3} c_{1}+a_{3} b_{1} c_{2}-a_{3} b_{2} c_{1}}\right) .
\end{aligned}
$$

That way you readily get the answer

$$
\left(x_{1}, x_{2}, x_{3}\right)=\left(\frac{47}{23}, \frac{27}{23},-\frac{11}{23}\right)
$$

Here, a part of the calculation is the denominators of the three fractions, namely, the determinant of $\left[\begin{array}{ccc}2 & -1 & 4 \\ 1 & 2 & 5 \\ 3 & -1 & 2\end{array}\right]$. It is -23 :

$$
\begin{aligned}
& \text { The determinant of }\left[\begin{array}{ccc}
2 & -1 & 4 \\
1 & 2 & 5 \\
3 & -1 & 2
\end{array}\right] \\
& =a_{1} b_{2} c_{3}-a_{1} b_{3} c_{2}-a_{2} b_{1} c_{3}+a_{2} b_{3} c_{1}+a_{3} b_{1} c_{2}-a_{3} b_{2} c_{1} \\
& =2 \cdot 2 \cdot 2-2 \cdot 5 \cdot(-1)-(-1) \cdot 1 \cdot 2+(-1) \cdot 5 \cdot 3+4 \cdot 1 \cdot(-1)-4 \cdot 2 \cdot 3 \\
& =8-(-10)-(-2)+(-15)+(-4)-24 \\
& =-23 .
\end{aligned}
$$

Then you will also have to calculate the numerators of the three fractions as well, in order to get $x_{1}, x_{2}$ and $x_{3}$. I know it takes a while to do all the work, but this is a sure thing way, as in you will get the correct answer as long as you carefully sub all the numbers, and then crunch the numbers correctly.

Exercise 2. Solve ( $* *$ ) on page 5 using Formula on page 6, following the above guidelines.

Now, if you followed this lecture up until this point, two things will come to your mind:

Question 1. In the above example, the calculation was quite involved due to the complexity of the formula. Is there another, simpler formula?

- My short answer is 'no'.

Then you would ask:

Question 2. If that's the case, is there a method that algorithmically deduces the answer, for each concrete system of equations $(3 \times 3)$, something like $(* *)$ on page 5 , without ever relying on the formula?

- My short answer is 'yes', though today we won't address it. The name for it is Gaussian elimination method . Today we won't talk about this, though we are going to spend a good deal of time on it in the upcoming weeks.

Now, that's not the end of the story. Be that as it may, there is an interesting twist. The 'Gaussian elimination method', as I just mentioned above, is actually tailor-made for calculating the determinants (of a matrix filled by numbers). What's more: When the Gaussian elimination method is being employed in a different context than evaluating a determinant, the gist of the process actually ends up being essentially the same as evaluating a determinant, in a certain justifiable sense which I won't elaborate.

So, the bottom line is, the notion of determinants is absolutely indispensable. Like I said before, you cannot really get away from the determinants. You should never get turned off by the determinants despite their complex appearances. For a good reason, you need to thoroughly investigate the determinants. Determinants are central in linear algebra, and are one of the few core conepts in all math for that matter. It is no exaggeration to say that determinants have a bearing on just about every aspect of math.

Before wrapping up, I want to just show you the $4 \times 4$ case, the shape of the determinant of the $4 \times 4$ matrix $\left[\begin{array}{cccc}a_{1} & a_{2} & a_{3} & a_{4} \\ b_{1} & b_{2} & b_{3} & b_{4} \\ c_{1} & c_{2} & c_{3} & c_{4} \\ d_{1} & d_{2} & d_{3} & d_{4}\end{array}\right]: \quad$ Have a glimpse at it:

$$
\begin{aligned}
& \text { The determinant of }\left[\begin{array}{llll}
a_{1} & a_{2} & a_{3} & a_{4} \\
b_{1} & b_{2} & b_{3} & b_{4} \\
c_{1} & c_{2} & c_{3} & c_{4} \\
d_{1} & d_{2} & d_{3} & d_{4}
\end{array}\right] \\
& =a_{1} b_{2} c_{3} d_{4}-a_{1} b_{2} c_{4} d_{3}-a_{1} b_{3} c_{2} d_{4}+a_{1} b_{3} c_{4} d_{2}+a_{1} b_{4} c_{2} d_{3}-a_{1} b_{4} c_{3} d_{2} \\
& \quad-a_{2} b_{1} c_{3} d_{4}+a_{2} b_{1} c_{4} d_{3}+a_{2} b_{3} c_{1} d_{4}-a_{2} b_{3} c_{4} d_{1}-a_{2} b_{4} c_{1} d_{3}+a_{2} b_{4} c_{3} d_{1} \\
& \quad+a_{3} b_{1} c_{2} d_{4}-a_{3} b_{1} c_{4} d_{2}-a_{3} b_{2} c_{1} d_{4}+a_{3} b_{2} c_{4} d_{1}+a_{3} b_{4} c_{1} d_{2}-a_{3} b_{4} c_{2} d_{1} \\
& \quad-a_{4} b_{1} c_{2} d_{3}+a_{4} b_{1} c_{3} d_{2}+a_{4} b_{2} c_{1} d_{3}-a_{4} b_{2} c_{3} d_{1}-a_{4} b_{3} c_{1} d_{2}+a_{4} b_{3} c_{2} d_{1}
\end{aligned}
$$

