Math 105 TOPICS IN MATHEMATICS

STUDY GUIDE FOR FINAL EXAM – FC

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- **§31. Trigonometry – II.**

'Radians' and 'degrees' are two alternative units to measure angles. We are familiar with 'degrees'. Below is the basic special values of 'cos' and 'sin' which you need to remember:

- **Basic 'cos' and 'sin' values – I.** $\mathbf{0^\circ \leq \theta \leq 90^\circ}$**.**

$$\cos\ 0^\circ = 1, \qquad \sin\ 0^\circ = 0,$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \sin 30^\circ = \frac{1}{2},$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} \ \left(= \frac{\sqrt{2}}{2}\right), \qquad \sin 45^\circ = \frac{1}{\sqrt{2}} \ \left(= \frac{\sqrt{2}}{2}\right),$$

$$\cos 60^\circ = \frac{1}{2}, \qquad \sin 60^\circ = \frac{\sqrt{3}}{2},$$

$$\cos 90^\circ = 0, \qquad \sin 90^\circ = 1.$$

- **Basic 'cos' and 'sin' values – II.** $\mathbf{90^\circ < \theta \leq 180^\circ}$**.**

$$\cos 120^\circ = -\frac{1}{2}, \qquad \sin 120^\circ = \frac{\sqrt{3}}{2},$$

$$\cos 135^\circ = -\frac{1}{\sqrt{2}} \ \left(= -\frac{\sqrt{2}}{2}\right), \qquad \sin 135^\circ = \frac{1}{\sqrt{2}} \ \left(= \frac{\sqrt{2}}{2}\right),$$

$$\cos 150^\circ = -\frac{\sqrt{3}}{2}, \qquad \sin 150^\circ = \frac{1}{2},$$

$$\cos 180^\circ = -1, \qquad \sin 180^\circ = 0.$$

- **Basic 'cos' and 'sin' values – III.** $\boldsymbol{-90° \le \theta < 0°}$.

$$\cos\left(-30°\right) = \frac{\sqrt{3}}{2}, \qquad \sin\left(-30°\right) = -\frac{1}{2},$$

$$\cos\left(-45°\right) = \frac{1}{\sqrt{2}}\left(= \frac{\sqrt{2}}{2}\right), \qquad \sin\left(-45°\right) = -\frac{1}{\sqrt{2}}\left(= -\frac{\sqrt{2}}{2}\right),$$

$$\cos\left(-60°\right) = \frac{1}{2}, \qquad \sin\left(-60°\right) = -\frac{\sqrt{3}}{2},$$

$$\cos\left(-90°\right) = 0, \qquad \sin\left(-90°\right) = -1.$$

- **Basic 'cos' and 'sin' values – IV.** $\boldsymbol{-180° < \theta < -90°}$.

$$\cos\left(-120°\right) = -\frac{1}{2}, \qquad \sin\left(-120°\right) = -\frac{\sqrt{3}}{2},$$

$$\cos\left(-135°\right) = -\frac{1}{\sqrt{2}}\left(= -\frac{\sqrt{2}}{2}\right), \qquad \sin\left(-135°\right) = -\frac{1}{\sqrt{2}}\left(= -\frac{\sqrt{2}}{2}\right),$$

$$\cos\left(-150°\right) = -\frac{\sqrt{3}}{2}, \qquad \sin\left(-150°\right) = -\frac{1}{2},$$

$$\cos\left(-180°\right) = -1, \qquad \sin\left(-180°\right) = 0.$$

- Now, check out the conversion table in the next page:

radian	degree
0	0°
$\frac{\pi}{6}$	30°
$\frac{\pi}{4}$	45°
$\frac{\pi}{3}$	60°
$\frac{\pi}{2}$	90°
$\frac{2\pi}{3}$	120°
$\frac{3\pi}{4}$	135°
$\frac{5\pi}{6}$	150°
π	180°
$-\frac{\pi}{6}$	-30°
$-\frac{\pi}{4}$	-45°
$-\frac{\pi}{3}$	-60°
$-\frac{\pi}{2}$	-90°
$-\frac{2\pi}{3}$	-120°
$-\frac{3\pi}{4}$	-135°
$-\frac{5\pi}{6}$	-150°
$-\pi$	-180°

This allows us to rewritre everything using radians:

- **Basic 'cos' and 'sin' values (in radians) – I.** $\boldsymbol{0 \le \theta \le \frac{\pi}{2}}$**.**

$$\cos\ 0 = 1, \qquad \sin\ 0 = 0,$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \qquad \sin\frac{\pi}{6} = \frac{1}{2},$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \left(= \frac{\sqrt{2}}{2}\right), \qquad \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \left(= \frac{\sqrt{2}}{2}\right),$$

$$\cos\frac{\pi}{3} = \frac{1}{2}, \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},$$

$$\cos\frac{\pi}{2} = 0, \qquad \sin\frac{\pi}{2} = 1.$$

- **Basic 'cos' and 'sin' values (in radians) – II.** $\boldsymbol{\frac{\pi}{2} < \theta \le \pi}$**.**

$$\cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2},$$

$$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} \left(= -\frac{\sqrt{2}}{2}\right), \qquad \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}} \left(= \frac{\sqrt{2}}{2}\right),$$

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}, \qquad \sin\frac{5\pi}{6} = \frac{1}{2},$$

$$\cos\ \pi = -1. \qquad \sin\ \pi = 0.$$

- **Basic 'cos' and 'sin' values (in radians) – III.** $\boldsymbol{-\frac{\pi}{2} \leq \theta < 0.}$

$$\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \qquad \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2},$$

$$\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(=\frac{\sqrt{2}}{2}\right), \qquad \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\left(=-\frac{\sqrt{2}}{2}\right),$$

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}, \qquad \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2},$$

$$\cos\left(-\frac{\pi}{2}\right) = 0, \qquad \sin\left(-\frac{\pi}{2}\right) = -1.$$

- **Basic 'cos' and 'sin' values (in radians) – IV.** $\boldsymbol{-\pi \leq \theta < \frac{\pi}{2}.}$

$$\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2}, \qquad \sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2},$$

$$\cos\left(-\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(=\frac{\sqrt{2}}{2}\right), \qquad \sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}\left(=-\frac{\sqrt{2}}{2}\right),$$

$$\cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \qquad \sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2},$$

$$\cos\left(-\pi\right) = -1, \qquad \sin\left(-\pi\right) = 0.$$

- **§32. Distance.**

Recall the following definition:

Definition. Let P and Q be two points, both lying in the xy-coordiate system. Suppose the coordinate reading of P and Q are given by

$$P = \left(a,\ b\right), \qquad \text{and} \qquad Q = \left(c,\ d\right),$$

respectively. Then the distance $\left|PQ\right|$ between P and Q is

$$\boxed{\sqrt{\left(c - a\right)^2 + \left(d - b\right)^2}} \quad .$$

⋆ The following is a speial case:

Let P be a point, lying in the xy-coordiate system. Suppose the coordinate reading of P is given by

$$P = \left(a,\ b\right).$$

Meanwhile, let O be the coordinate origin:

$$O = \left(0,\ 0\right).$$

Then the distance $\left|PO\right|$ between P and O is

$$\boxed{\sqrt{a^2 + b^2}} \quad .$$

Example. Let's find the distance between

$$P = \Big(3,\ 5\Big), \qquad\qquad Q = \Big(4,\ 7\Big).$$

By definition,

$$\begin{aligned} |\,PQ\,| &= \sqrt{\Big(4 - 3\Big)^2 + \Big(7 - 5\Big)^2} \\ &= \sqrt{\ 1^2 + 2^2\ } \\ &= \sqrt{5}\,. \end{aligned}$$

Example. Let's find the distance between

$$P = \Big(1,\ -3\Big), \qquad\qquad Q = \Big(-2,\ 4\Big).$$

By definition,

$$\begin{aligned} |\,PQ\,| &= \sqrt{\Big(-2 - 1\Big)^2 + \Big(4 - (-3)\Big)^2} \\ &= \sqrt{\ (-3)^2 + 7^2\ } \\ &= \sqrt{\ 9 + 49\ } \\ &= \sqrt{58}\,. \end{aligned}$$

Example. Let's find the distance between

$$P = \Big(-5,\ 0\Big), \qquad Q = \Big(6,\ 0\Big).$$

By definition,

$$\begin{aligned} |PQ| &= \sqrt{\Big(6 - (-5)\Big)^2 + \Big(0 - 0\Big)^2} \\ &= \sqrt{11^2 + 0^2} \\ &= \sqrt{11^2} \\ &= 11. \end{aligned}$$

Example. Let's find the distance between

$$P = \Big(12,\ 1\Big), \qquad O = \Big(0,\ 0\Big).$$

By definition,

$$\begin{aligned} |PO| &= \sqrt{12^2 + 1^2} \\ &= \sqrt{144 + 1} \\ &= \sqrt{145}. \end{aligned}$$

Q. Find the distance between P and Q.

(1) $P = \left(2,\ 0\right), \qquad Q = \left(3,\ 1\right).$

(2) $P = \left(-4,\ 3\right), \qquad Q = \left(-1,\ 7\right).$

(3) $P = \left(-3,\ 4\right), \qquad Q = \left(2,\ 16\right).$

(4) $P = \left(100,\ 0\right), \qquad Q = \left(100,\ 1\right).$

(5) $P = \left(4,\ 1\right), \qquad O = \left(0,\ 0\right).$

(6) $O = \left(0,\ 0\right), \qquad Q = \left(8,\ 15\right).$

[**Answers**]**:**

(1) $\sqrt{2}$. (2) 5. (3) 13. (4) 1.

(5) $\sqrt{17}$. (6) 17.

- **Most basic property of sin and cos.**

No matter what you do, please remember the following, which is extremely important:

$$\boxed{\left(\cos\theta\right)^2 + \left(\sin\theta\right)^2 = 1} .$$

This is true no matter what θ is.

Example. $$\left(\cos\frac{\pi}{9}\right)^2 + \left(\sin\frac{\pi}{9}\right)^2 \;=\; 1.$$

Example. $$\left(\cos\frac{\pi}{5}\right)^2 + \left(\sin\frac{\pi}{5}\right)^2 \;=\; 1.$$

Example. $$\left(\cos\frac{3\pi}{7}\right)^2 + \left(\sin\frac{3\pi}{7}\right)^2 \;=\; 1.$$

Example. $$\left(\cos\frac{\pi}{\sqrt{2}}\right)^2 + \left(\sin\frac{\pi}{\sqrt{2}}\right)^2 \;=\; 1.$$

Example. $$\left(\cos\left(-\frac{\pi}{15}\right)\right)^2 + \left(\sin\left(-\frac{\pi}{15}\right)\right)^2 \;=\; 1.$$

⋆ Can you paraphrase the above identity in terms of the distance? I bet you can.

" The distance between

$$P \;=\; \Big(\cos\theta,\;\sin\theta\Big)$$

and the coordinate origin $O \;=\; \Big(0,\;0\Big)$ is always 1. "

⋆ Here is a further paraphrase:

" The point

$$P \;=\; \Big(\cos\theta,\;\sin\theta\Big)$$

always lies in *the unit circle*, the circle with radius 1 centered at the origin."

Here is one important exercise:

Q. Find the distance between

$$P = \left(\cos\frac{\pi}{4},\ \sin\frac{\pi}{4}\right), \qquad Q = \left(\cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right).$$

Solution: First, recall

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \qquad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2},$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \qquad \sin\frac{\pi}{6} = \frac{1}{2}.$$

Thus, by definition,

$$|PQ| = \sqrt{\left(\left(\cos\frac{\pi}{4}\right) - \left(\cos\frac{\pi}{6}\right)\right)^2 + \left(\left(\sin\frac{\pi}{4}\right) - \left(\sin\frac{\pi}{6}\right)\right)^2}$$

$$= \sqrt{\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)^2}$$

$$= \sqrt{\left(\frac{\sqrt{3} - \sqrt{2}}{2}\right)^2 + \left(\frac{1 - \sqrt{2}}{2}\right)^2}$$

$$= \sqrt{\frac{3 + 2 - 2\sqrt{3}\cdot\sqrt{2}}{4} + \frac{1 + 2 - 2\cdot 1\cdot\sqrt{2}}{4}}$$

$$= \sqrt{\frac{5 - 2\sqrt{6}}{4} + \frac{3 - 2\sqrt{2}}{4}}$$

$$= \sqrt{\frac{5 - 2\sqrt{6} + 3 - 2\sqrt{2}}{4}}$$

$$= \sqrt{\frac{8 - 2\sqrt{6} - 2\sqrt{2}}{4}} \quad \left(= \frac{1}{2}\sqrt{8 - 2\sqrt{6} - 2\sqrt{2}}\right).$$

Q. Let P and Q be as in the previous Q. Let

$$R = \left(\cos\frac{\pi}{12},\ \sin\frac{\pi}{12}\right), \qquad S = \big(1,\ 0\big).$$

Explain why $\left|PQ\right|$ and $\left|RS\right|$ are equal. Then use that fact and the result of the previous Q to evaluate $\cos\frac{\pi}{12}$.

Solution:

P and Q are both lying in the unit circle such that the angle $\angle POQ$ equals $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$. R and S are also both lying in the unit circle such that the angle $\angle ROS$ equals $\frac{\pi}{12} - 0 = \frac{\pi}{12}$. Hence $\left|PQ\right|$ and $\left|RS\right|$ are naturally equal. Now,

$$\begin{aligned}\left|RS\right| &= \sqrt{\left(\left(\cos\frac{\pi}{12}\right) - 1\right)^2 + \left(\sin\frac{\pi}{12}\right)^2} \\ &= \sqrt{\left(\cos\frac{\pi}{12}\right)^2 - 2\left(\cos\frac{\pi}{12}\right) + 1 + \left(\sin\frac{\pi}{12}\right)^2} \\ &= \sqrt{1 - 2\left(\cos\frac{\pi}{12}\right) + 1} \\ &= \sqrt{2 - 2\left(\cos\frac{\pi}{12}\right)}\,.\end{aligned}$$

This equals

$$\left|RS\right| = \sqrt{\frac{8 - 2\sqrt{6} - 2\sqrt{2}}{4}}\,.$$

So

$$2 \,-\, 2\left(\cos\,\frac{\pi}{12}\right) \;=\; \frac{8 \,-\, 2\sqrt{6} \,-\, 2\sqrt{2}}{4}.$$

Solve this for $\cos\frac{\pi}{12}$:

$$2\,\cos\frac{\pi}{12} \;=\; \frac{2\sqrt{6} \,+\, 2\sqrt{2}}{4}$$

$$\Longrightarrow \qquad \cos\frac{\pi}{12} \;=\; \frac{\sqrt{6} \,+\, \sqrt{2}}{4}.$$

• §34. Mean values.

Recall

(1) the mean of a_1 is $\dfrac{a_1}{1}$.

(2) the mean of a_1 and a_2 is $\dfrac{a_1 + a_2}{2}$.

(3) the mean of a_1, a_2 and a_3 is $\dfrac{a_1 + a_2 + a_3}{3}$.

(4) the mean of a_1, a_2, a_3 and a_4 is $\dfrac{a_1 + a_2 + a_3 + a_4}{4}$.

(5) the mean of a_1, a_2, a_3, a_4 and a_5 is $\dfrac{a_1 + a_2 + a_3 + a_4 + a_5}{5}$.

$\vdots$

Q. (1) Find the mean of 2.

(2) Find the mean of 3 and –7.

(3) Find the mean of 3, 5 and 17.

(4) Find the mean of 4, 6, 8 and 10.

(5) Find the mean of –1, 2, –3, 4 and –5.

(6) Find the mean of 96, 48, 24, 12, 6 and 3.

[**Answers**]: (1) 2. (2) –2. (3) $\dfrac{25}{3}$. (4) 7.

(5) $-\dfrac{3}{5}$. (6) $\dfrac{63}{2}$.

Example. Let $f(x) = x^2$. Then

(1) the mean of $f(1)$ is $\dfrac{1^2}{1}$.

(2) the mean of $f(1)$ and $f(2)$ is $\dfrac{1^2+2^2}{2}$.

(3) the mean of $f(1)$, $f(2)$ and $f(3)$ is $\dfrac{1^2+2^2+3^2}{3}$.

(4) the mean of $f(1)$, $f(2)$, $f(3)$ and $f(4)$ is $\dfrac{1^2+2^2+3^2+4^2}{4}$.

(5) the mean of $f(1)$, $f(2)$, $f(3)$, $f(4)$ and $f(5)$ is $\dfrac{1^2+2^2+3^2+4^2+5^2}{5}$.

⋮

Example. Let $f(x) = x^2$ again. Then

- the mean of $f\left(\frac{x}{2}\right)$ and $f\left(\frac{2x}{2}\right)$ is $\dfrac{\left(\frac{x}{2}\right)^2+\left(\frac{2x}{2}\right)^2}{2}$,
- the mean of $f\left(\frac{x}{3}\right)$, $f\left(\frac{2x}{3}\right)$ and $f\left(\frac{3x}{3}\right)$ is $\dfrac{\left(\frac{x}{3}\right)^2+\left(\frac{2x}{3}\right)^2+\left(\frac{3x}{3}\right)^2}{3}$,
- the mean of $f\left(\frac{x}{4}\right)$, $f\left(\frac{2x}{4}\right)$, $f\left(\frac{3x}{4}\right)$ and $f\left(\frac{4x}{4}\right)$ is

$$\frac{\left(\frac{x}{4}\right)^2+\left(\frac{2x}{4}\right)^2+\left(\frac{3x}{4}\right)^2+\left(\frac{4x}{4}\right)^2}{4},$$

⋮

Here,

$$\frac{\left(\frac{x}{2}\right)^2 + \left(\frac{2x}{2}\right)^2}{2} = \left(1^2 + 2^2\right)\frac{x^2}{2^3},$$

$$\frac{\left(\frac{x}{3}\right)^2 + \left(\frac{2x}{3}\right)^2 + \left(\frac{3x}{3}\right)^2}{3} = \left(1^2 + 2^2 + 3^2\right)\frac{x^2}{3^3},$$

$$\frac{\left(\frac{x}{4}\right)^2 + \left(\frac{2x}{4}\right)^2 + \left(\frac{3x}{4}\right)^2 + \left(\frac{4x}{4}\right)^2}{4} = \left(1^2 + 2^2 + 3^2 + 4^2\right)\frac{x^2}{4^3},$$

$$\vdots$$

As you extrapolate the patterns, you agree that the following is true:

- the mean of $f\left(\frac{x}{n}\right),\ f\left(\frac{2x}{n}\right),\ f\left(\frac{3x}{n}\right),\ \cdots\ f\left(\frac{nx}{n}\right)$ is

$$\left(\underline{1^2 + 2^2 + 3^2 + \cdots + n^2}\right)\frac{x^2}{n^3}.$$

As for the underlined part, let's recall the formula:

Formula. (from "Review of Lectures – XXVII").

$$\boxed{1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n}.$$

So the above is paraphrased as follows:

- the mean of $f\Big(\dfrac{x}{n}\Big)$, $f\Big(\dfrac{2x}{n}\Big)$, $f\Big(\dfrac{3x}{n}\Big)$, $\cdots$, $f\Big(\dfrac{nx}{n}\Big)$ is

$$\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right)\frac{x^2}{n^3},$$

that is,

$$\left(\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{n} + \frac{1}{6}\cdot\frac{1}{n^2}\right)x^2.$$

It is useful for later purpose to examine the state of this quantity when n grows arbitrarily large. When n grows, like

$$\begin{aligned} n &= 1000, \\ n &= 1000000, \\ n &= 1000000000, \\ &\ \vdots \end{aligned}$$

then accordingly $\dfrac{1}{n}$ and $\dfrac{1}{n^2}$ become negligible:

$$\begin{array}{lcll} n = 1000 & \Longrightarrow & \dfrac{1}{n} = 0.001, & \dfrac{1}{n^2} = 0.000001, \\ n = 1000000 & \Longrightarrow & \dfrac{1}{n} = 0.000001, & \dfrac{1}{n^2} = 0.000000000001, \\ n = 1000000000 & \Longrightarrow & \dfrac{1}{n} = 0.000000001, & \dfrac{1}{n^2} = 0.000000000000000001, \\ & \vdots & & \end{array}$$

More precisely,

$$\lim_{n \to \infty} \frac{1}{n} = 0, \qquad \lim_{n \to \infty} \frac{1}{n^2} = 0.$$

Hence

$$\lim_{n \to \infty} \left(\frac{1}{3} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{6} \cdot \frac{1}{n^2} \right) x^2 = \frac{1}{3} x^2.$$

- **Summary.**

" *For*

$$f(x) = x^2,$$

the mean of $f\left(\frac{x}{n}\right), f\left(\frac{2x}{n}\right), f\left(\frac{3x}{n}\right), \cdots, f\left(\frac{nx}{n}\right),$ *as* $n \longrightarrow \infty,$

is

$$\frac{1}{3} x^2. \text{"}$$

Notation. For a given $f(x)$, the notation $M_n(f)(x)$ stands for the mean of

$$f\left(\frac{x}{n}\right), f\left(\frac{2x}{n}\right), f\left(\frac{3x}{n}\right), \cdots, f\left(\frac{nx}{n}\right).$$

Also, the notation $M(f)(x)$ stands for the limit

$$\lim_{n \to \infty} M_n(f)(x).$$

Results.

(1) $f(x) = x \implies M_n(f)\,(x) = \left(\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{n}\right)x,$

$$M\,(f)\,(x) = \frac{1}{2}x.$$

(2) $f(x) = x^2 \implies M_n(f)\,(x) = \left(\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{n} + \frac{1}{6}\cdot\frac{1}{n^2}\right)x^2,$

$$M\,(f)\,(x) = \frac{1}{3}x^2.$$

(3) $f(x) = x^3 \implies M_n(f)\,(x) = \left(\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{n} + \frac{1}{4}\cdot\frac{1}{n^2}\right)x^3,$

$$M\,(f)\,(x) = \frac{1}{4}x^3.$$

(4) $f(x) = x^4 \implies M_n(f)\,(x) = \left(\frac{1}{5} + \frac{1}{2}\cdot\frac{1}{n} + \frac{1}{3}\cdot\frac{1}{n^2} - \frac{1}{30}\cdot\frac{1}{n^4}\right)x^4,$

$$M\,(f)\,(x) = \frac{1}{5}x^4.$$

(5) $f(x) = x^5 \implies M_n(f)\,(x) = \left(\frac{1}{6} + \frac{1}{2}\cdot\frac{1}{n} + \frac{5}{12}\cdot\frac{1}{n^2} - \frac{1}{12}\cdot\frac{1}{n^4}\right)x^5,$

$$M\,(f)\,(x) = \frac{1}{6}x^5.$$

Q. Find $M_n(f)(x)$ and $M(f)(x)$ for

(6) $f(x) = x^6.$

[<u>**Answer**</u>]**:**

$$M_n(f)(x) = \left(\frac{1}{7} + \frac{1}{2}\cdot\frac{1}{n} + \frac{1}{2}\cdot\frac{1}{n^2} - \frac{1}{6}\cdot\frac{1}{n^4} + \frac{1}{42}\cdot\frac{1}{n^6}\right)x^6,$$

$$M(f)(x) = \frac{1}{7}x^6.$$

• §35. Trigonometry – IV. Definite integrals.

Axiom 1.
$$\cos\left(\theta-\phi\right) = \left(\cos\theta\right)\left(\cos\phi\right) + \left(\sin\theta\right)\left(\sin\phi\right).$$

Axiom 2.
$$\sin\left(\theta-\phi\right) = \left(\sin\theta\right)\left(\cos\phi\right) - \left(\cos\theta\right)\left(\sin\phi\right).$$

Axiom 3.
$$\cos\left(\theta+\phi\right) = \left(\cos\theta\right)\left(\cos\phi\right) - \left(\sin\theta\right)\left(\sin\phi\right).$$

Axiom 4.
$$\sin\left(\theta+\phi\right) = \left(\sin\theta\right)\left(\cos\phi\right) + \left(\cos\theta\right)\left(\sin\phi\right).$$

Double angle formula for cos.

$$\cos\left(2\theta\right) = \left(\cos\theta\right)^2 - \left(\sin\theta\right)^2.$$

Double angle formula for cos – version 2.

$$\cos\left(2\theta\right) = 2\left(\cos\theta\right)^2 - 1.$$

Double angle formula for cos – version 3.

$$\cos\left(2\theta\right) = 1 - 2\left(\sin\theta\right)^2.$$

Double angle formula for sin.

$$\boxed{\sin\Big(2\theta\Big) \;=\; 2\Big(\cos\theta\Big)\Big(\sin\theta\Big)}\,.$$

Formula A. $$\boxed{2\Big(\cos\theta\Big)\Big(\cos\phi\Big) \;=\; \cos\Big(\theta-\phi\Big) + \cos\Big(\theta+\phi\Big)}\,.$$

Formula B. $$\boxed{2\Big(\sin\theta\Big)\Big(\sin\phi\Big) \;=\; \cos\Big(\theta-\phi\Big) - \cos\Big(\theta+\phi\Big)}\,.$$

Formula C. $$\boxed{2\Big(\sin\theta\Big)\Big(\cos\phi\Big) \;=\; \sin\Big(\theta-\phi\Big) + \sin\Big(\theta+\phi\Big)}\,.$$

Formula D. $$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} \;=\; 1.}\,.$$

Notation. For a given $f(x)$, the notation $M_n(f)(x)$ stands for the mean of

$$f\Big(\frac{x}{n}\Big),\; f\Big(\frac{2x}{n}\Big),\; f\Big(\frac{3x}{n}\Big),\; \cdots,\; f\Big(\frac{nx}{n}\Big).$$

Also, the notation $M(f)(x)$ stands for the limit

$$\lim_{n\to\infty} M_n(f)(x).$$

Formula. Let n be a positive integer; $n = 1, 2, 3, 4, \cdots$. Let

$$f(x) \;=\; x^n.$$

Then we have

$$\boxed{M(f)(x) = \frac{1}{n+1}x^n}\,.$$

Fact. For

$$f(x) = \cos x,$$

we have

$$\begin{cases} M_n(f)(x) = \dfrac{\left(\sin \dfrac{x}{2}\right)\left(\cos\left(\dfrac{x}{2} + \dfrac{x}{2n}\right)\right)}{n\left(\sin \dfrac{x}{2n}\right)}, \\ \\ M(f)(x) = \dfrac{\sin x}{x}. \end{cases}$$

Fact. For

$$g(x) = \sin x,$$

we have

$$\begin{cases} M_n(g)(x) = \dfrac{\left(\sin \dfrac{x}{2}\right)\left(\sin\left(\dfrac{x}{2} + \dfrac{x}{2n}\right)\right)}{n\left(\sin \dfrac{x}{2n}\right)}, \\ \\ M(g)(x) = \dfrac{1 - \cos x}{x}. \end{cases}$$

- **Definite integrals.**

Definition. The <u>definite integral</u> of $f\left(t\right)$ over the interval $\left[0,\ x\right]$ is simply

$$\int_{t=0}^{x} f\left(t\right) dt \;=\; x \cdot M\left(f\right)\left(x\right).$$

If you apply this definition, then we immediately get

Formula. Let n be a positive integer; $n = 1, 2, 3, 4, \cdots$. Then

$$\int_{t=0}^{x} t^{n}\ dt \;=\; \frac{1}{n+1}\, x^{n+1}.$$

Formula.

$$\int_{t=0}^{x} \cos t\ dt \;=\; \sin x, \qquad \int_{t=0}^{x} \sin t\ dt \;=\; 1 - \cos x.$$

Example. $$\int_{t=0}^{1} t^2 \ dt \ = \ \frac{1}{2+1} 1^{2+1} \ = \ 3.$$

Example. $$\int_{t=0}^{2} t^4 \ dt \ = \ \frac{1}{4+1} 2^{4+1} \ = \ \frac{32}{5}.$$

Example. $$\int_{t=0}^{\frac{\pi}{2}} \cos t \ dt \ = \ \sin \frac{\pi}{2} \ = \ 1.$$

Example. $$\int_{t=0}^{\frac{\pi}{3}} \cos t \ dt \ = \ \sin \frac{\pi}{3} \ = \ \frac{\sqrt{3}}{2}.$$

Example. $$\int_{t=0}^{\frac{\pi}{2}} \sin t \ dt \ = \ 1 - \cos \frac{\pi}{2} \ = \ 1.$$

Example. $$\int_{t=0}^{\frac{\pi}{3}} \sin t \ dt \ = \ 1 - \cos \frac{\pi}{3} \ = \ \frac{1}{2}.$$

Q. Evaluate

(1) $\displaystyle\int_{t=0}^{4} t \ dt.$

(2) $\displaystyle\int_{t=0}^{7} t^3 \ dt.$

(3) $\displaystyle\int_{t=0}^{\frac{\pi}{4}} \cos t \ dt.$

(4) $\displaystyle\int_{t=0}^{\frac{\pi}{6}} \cos t \ dt.$

(5) $\displaystyle\int_{t=0}^{\frac{\pi}{4}} \sin t \ dt.$

(6) $\displaystyle\int_{t=0}^{\frac{\pi}{6}} \sin t \ dt.$

[**Answers**]**:** (1) 8. (2) $\dfrac{2401}{4}$. (3) $\dfrac{1}{\sqrt{2}}$.

(4) $\dfrac{1}{2}$. (5) $1 - \dfrac{1}{\sqrt{2}}$. (6) $1 - \dfrac{\sqrt{3}}{2}$.

- Recall

Fundamental theorem.

Suppose an antiderivative of $f(t)$ is $F(t)$. Then

$$\boxed{\int_{t=y}^{x} f(t)\,dt \;=\; F(x) - F(y).}$$

Notation. It is convenient to write the above theorem as

$$\boxed{\int_{t=y}^{x} f(t)\,dt \;=\; \Big[\, F(t) \,\Big]_{t=y}^{x}.}$$

Here, $\Big[\, F(t) \,\Big]_{t=y}^{x}$ simply means $F(x) - F(y)$.

Let's use some examples to illustrate how the evaluation goes:

Example.

$$\int_{t=2}^{3} t^2\,dt \;=\; \left[\, \frac{1}{3} t^3 \,\right]_{t=2}^{3}$$

$$= \frac{1}{3} 3^3 - \frac{1}{3} 2^3 = \frac{19}{3}.$$

Example.

$$\int_{t=-1}^{1} t^6 \; dt \;=\; \Big[\, \frac{1}{7} t^7 \,\Big]_{t=-1}^{1}$$

$$=\; \frac{1}{7} 1^7 \;-\; \frac{1}{7} \big(-1 \big)^7$$

$$=\; \frac{2}{7}.$$

• The following is inferred by what we have worked out today and 'Fundamental Theorem' above.

Quick Facts.

(1) An antiderivative of $\boxed{\cos x}$ is $\boxed{\sin x}$.

(2) An antiderivative of $\boxed{\sin x}$ is $\boxed{-\cos x}$.

Example.

$$\int_{t=\frac{\pi}{6}}^{\frac{\pi}{2}} \cos t \; dt \;=\; \Big[\, \sin t \,\Big]_{t=\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$=\; \sin \frac{\pi}{2} \;-\; \sin \frac{\pi}{6}$$

$$=\; 1 - \frac{1}{2} \;=\; \frac{1}{2}.$$

Example.

$$\int_{t=-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin t \; dt \;=\; \Big[\, -\cos t \,\Big]_{t=-\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=\; \Big(-\cos \frac{\pi}{2} \Big) \;-\; \Big(-\cos \Big(-\frac{\pi}{4} \Big) \Big)$$

$$=\; 0 - \Big(-\frac{1}{\sqrt{2}} \Big) \;=\; \frac{1}{\sqrt{2}}.$$

Q. Evaluate

(1) $\displaystyle\int_{t=-2}^{1} t^2 \; dt.$ (2) $\displaystyle\int_{t=1}^{\frac{3}{2}} t^5 \; dt.$

(3) $\displaystyle\int_{t=\frac{\pi}{4}}^{\frac{\pi}{3}} \cos t \; dt.$ (4) $\displaystyle\int_{t=-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos t \; dt.$

(5) $\displaystyle\int_{t=\frac{\pi}{6}}^{\frac{\pi}{4}} \sin t \; dt.$ (6) $\displaystyle\int_{t=-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin t \; dt.$

[**Answers**]**:** (1) 3. (2) $\dfrac{665}{384}.$ (3) $\dfrac{\sqrt{3}-\sqrt{2}}{2}.$

(4) $\dfrac{3}{2}.$ (5) $\dfrac{\sqrt{3}-\sqrt{2}}{2}.$ (6) $\dfrac{1}{\sqrt{2}}.$

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Example.

$$\int_{t=0}^{1} \left(t^3 + t^2\right) dt = \left(\int_{t=0}^{1} t^3 \, dt \right) + \left(\int_{t=0}^{1} t^2 \, dt \right)$$

$$= \left[\frac{1}{4} t^4 \right]_{t=0}^{1} + \left[\frac{1}{3} t^3 \right]_{t=0}^{1}$$

$$= \frac{1}{4} \cdot 1^4 + \frac{1}{3} \cdot 1^3 = \frac{7}{12}.$$

⋆ You can do it the following way:

$$\int_{t=0}^{1} \left(t^3 + t^2\right) dt = \left[\frac{1}{4} t^4 + \frac{1}{3} t^3 \right]_{t=0}^{1}$$

$$= \frac{1}{4} \cdot 1^4 + \frac{1}{3} \cdot 1^3 = \frac{7}{12}.$$

Example.

$$\int_{t=-1}^{4} \left(t^4 + 1\right) dt = \left(\int_{t=-1}^{4} t^4 \, dt \right) + \left(\int_{t=-1}^{4} 1 \, dt \right)$$

$$= \left[\frac{1}{5} t^5 \right]_{t=-1}^{4} + \left[t \right]_{t=-1}^{4}$$

$$= \left(\frac{1}{5} \cdot 4^5 - \frac{1}{5} \cdot \left(-1\right)^5 \right) + \left(4 - \left(-1\right)\right)$$

$$= 210.$$

⋆ You can do it the following way:

$$\begin{aligned}\int_{t=-1}^{4} \left(t^4 + 1\right)\, dt &= \left[\frac{1}{5}t^5 + t\right]_{t=-1}^{4} \\ &= \left(\frac{1}{5}\cdot 4^5 + 4\right) - \left(\frac{1}{5}\cdot\left(-1\right)^5 + \left(-1\right)\right) \\ &= 210.\end{aligned}$$

Example.

$$\begin{aligned}\int_{t=0}^{2} 2t^3\, dt &= 2\left(\int_{t=0}^{2} t^3\, dt\right) \\ &= 2\cdot\left[\frac{1}{4}t^4\right]_{t=0}^{2} \\ &= 2\cdot\left(\frac{1}{4}\cdot 2^4\right) \\ &= 8.\end{aligned}$$

⋆ You can do it the following way:

$$\begin{aligned}\int_{t=0}^{2} 2t^3\, dt &= \left[\frac{1}{2}t^4\right]_{t=0}^{2} \\ &= \frac{1}{2}\cdot 2^4 \\ &= 8.\end{aligned}$$

Example. $\displaystyle\int_{t=0}^{\frac{\pi}{3}} \Big(2\cos t \;+\; 3\sin t\Big)\, dt$

$$= \; 2\left(\int_{t=0}^{\frac{\pi}{3}} \cos t \; dt\right) \;+\; 3\left(\int_{t=0}^{\frac{\pi}{3}} \sin t \; dt\right)$$

$$= \; 2 \cdot \Big[\, \sin t \,\Big]_{t=0}^{\frac{\pi}{3}} \;+\; 3 \cdot \Big[\, -\cos t \,\Big]_{t=0}^{\frac{\pi}{3}}$$

$$= \; 2 \cdot \left(\left(\sin \frac{\pi}{3}\right) - \Big(\sin 0\Big)\right) + 3 \cdot \left(\left(-\cos \frac{\pi}{3}\right) - \Big(-\cos 0\Big)\right)$$

$$= \; 2 \cdot \left(\frac{\sqrt{3}}{2} - 0\right) + 3 \cdot \left(\left(-\frac{1}{2}\right) - \Big(-1\Big)\right)$$

$$= \; \sqrt{3} \;+\; \frac{3}{2}.$$

$\star$ You can do it the following way:

$$\int_{t=0}^{\frac{\pi}{3}} \Big(2\cos t \;+\; 3\sin t\Big)\, dt$$

$$= \; \Big[\, 2\sin t \;-\; 3\cos t \,\Big]_{t=0}^{\frac{\pi}{3}}$$

$$= \; \left(2\left(\sin \frac{\pi}{3}\right) - 3\left(\cos \frac{\pi}{3}\right)\right) - \left(2\Big(\sin 0\Big) - 3\Big(\cos 0\Big)\right)$$

$$= \; \left(2 \cdot \frac{\sqrt{3}}{2} - 3 \cdot \frac{1}{2}\right) - \Big(2 \cdot 0 \;-\; 3 \cdot 1\Big) \;=\; \sqrt{3} \;+\; \frac{3}{2}.$$

Q. Evaluate

(1) $\displaystyle\int_{t=3}^{5} \left(t^2 + 2t\right) dt.$ (2) $\displaystyle\int_{t=0}^{\frac{3}{2}} \left(4t^3 - 3t^2\right) dt.$

(3) $\displaystyle\int_{t=0}^{1} \left(t^3 - \frac{3}{2}t^2 + \frac{1}{2}t\right) dt.$

(4) $\displaystyle\int_{t=-1}^{0} \frac{\left(t+1\right)\left(t+2\right)\left(t+3\right)}{3!} dt.$

(5) $\displaystyle\int_{t=\frac{\pi}{6}}^{\frac{\pi}{3}} \left(3\cos t - 4\sin t\right) dt.$

(6) $\displaystyle\int_{t=0}^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right) dt.$

[**Answers**]: (1) $\dfrac{146}{3}$. (2) $\dfrac{27}{16}$. (3) $\dfrac{9}{64}$.

(4) $\dfrac{3}{8}$. (5) $\dfrac{1 - \sqrt{3}}{2}$. (6) $\sqrt{2}$.

- Next, I show you something which looks innocuous but is extremely important:

Important Formula.

$$\boxed{\frac{d}{dx}\cos x \;=\; -\sin x}\quad , \qquad \text{and} \qquad \boxed{\frac{d}{dx}\sin x \;=\; \cos x}\quad .$$

- **§37. Trigonometry – VI.**

Last topic:

"congruence."

We say

- 5 is congruent to 1 modulo 4, because $5 - 1 = 4$ is divisible by 4.
- 8 is congruent to 1 modulo 7, because $8 - 1 = 7$ is divisible by 7.
- 29 is congruent to 5 modulo 8, because $29 - 5 = 24$ is divisible by 8.
- 32 is congruent to 2 modulo 15, because $32 - 2 = 30$ is divisible by 15.
- 15 is congruent to 3 modulo 6, because $15 - 3 = 12$ is divisible by 6.
- 28 is congruent to 0 modulo 7, because $28 - 0 = 28$ is divisible by 7.

Write these as

$$5 \underset{4}{\equiv} 1, \qquad 8 \underset{7}{\equiv} 1, \qquad 29 \underset{8}{\equiv} 5,$$
$$32 \underset{15}{\equiv} 2, \qquad 15 \underset{6}{\equiv} 3, \qquad 28 \underset{7}{\equiv} 0.$$

- More generally, let a, b and r be integers, and $r \geq 2$. We say

"*a is congruent to b modulo r,*

if $a - b$ is divisible by r. We write

$$\boxed{a \underset{r}{\equiv} b}$$

."

Exercise 1. True or false:

(1) $22 \underset{5}{\equiv} 0.$ (2) $13 \underset{4}{\equiv} 3.$ (3) $100 \underset{25}{\equiv} 0.$

(4) $17 \underset{5}{\equiv} 2.$ (5) $64 \underset{20}{\equiv} 4.$ (6) $121 \underset{37}{\equiv} 10.$

[**Answers**]: (1) False. (2) False. (3) True.

(4) True. (5) True. (6) True.

Today I am going to rely on the notion of 'congruence modulo 4'. Agree that

- $0, 4, 8, 12, 16, 20, 24, 28, \cdots$ are all congruent to 0 modulo 4,
- $1, 5, 9, 13, 17, 21, 25, 29, \cdots$ are all congruent to 1 modulo 4,
- $2, 6, 10, 14, 18, 22, 26, 30, \cdots$ are all congruent to 2 modulo 4,
- $3, 7, 11, 15, 19, 23, 27, 31, \cdots$ are all congruent to 3 modulo 4.

- Recall the following: Let x be an arbitrary real number. Then

$$\boxed{\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots},$$

and

$$\boxed{\sin x = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots}.$$

I said these can be the definition of $\sin x$ and $\cos x$. Meanwhile, let's recall $\big($from "Review of Lectures – XVIII"$\big)$:

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

The patterns in those three look alike. Is there a relation between

$$e^x, \qquad \cos x \qquad \text{and} \qquad \sin x \ ?$$

If you just stay within the real numbers, you cannot find any relationship. As for this, remember that I briefly touched the subject of complex numbers, and more specifically the role of the unit imaginary number $\sqrt{-1}$, in "Review of Lectures – XV". An important role is played by not just $\sqrt{-1}$ alone, but its powers. When it comes to the powers, $\sqrt{-1}$ and -1 are in sync with each other. Let's recall:

- $\boldsymbol{\big(-1\big)}$**-to-the-powers.** We have

$$\begin{aligned}
\big(-1\big)^1 &= -1, \\
\big(-1\big)^2 &= 1, \\
\big(-1\big)^3 &= -1, \\
\big(-1\big)^4 &= 1, \\
\big(-1\big)^5 &= -1, \\
\big(-1\big)^6 &= 1, \\
\big(-1\big)^7 &= -1, \\
\big(-1\big)^8 &= 1, \\
&\ \vdots \qquad \vdots
\end{aligned}$$

In short,

$$\left(-1\right)^n \quad = \quad \begin{cases} \quad 1 & \quad \left(\text{if } n \text{ is } \underline{\underline{\text{even}}}\right), \\ \\ -1 & \quad \left(\text{if } n \text{ is } \underline{\underline{\text{odd}}}\right). \end{cases}$$

- **$\left(\sqrt{-1}\right)$-to-the-powers.** Here, we adopt the unit imaginary number

$$i \;=\; \sqrt{-1}.$$

This number is 'defined' as

$$i^2 \quad = \quad -1.$$

So,

$$\begin{aligned} i^1 &\quad = \quad i, \\ i^2 &\quad = \quad -1, \\ i^3 &\quad = \quad -i, \\ i^4 &\quad = \quad 1, \\ i^5 &\quad = \quad i, \\ i^6 &\quad = \quad -1, \\ i^7 &\quad = \quad -i, \\ i^8 &\quad = \quad 1, \\ i^9 &\quad = \quad i, \\ i^{10} &\quad = \quad -1, \\ i^{11} &\quad = \quad -i, \\ i^{12} &\quad = \quad 1, \\ &\quad \vdots \end{aligned}$$

In short,

$$i^n = \begin{cases} 1 & \left(\text{if } n \underset{4}{\equiv} 0\right), \\ i & \left(\text{if } n \underset{4}{\equiv} 1\right), \\ -1 & \left(\text{if } n \underset{4}{\equiv} 2\right), \\ -i & \left(\text{if } n \underset{4}{\equiv} 3\right). \end{cases}$$

With that in mind, why don't we substitute x with $i\,x$ in

$$\begin{aligned} e^x = {} & 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 \\ & + \frac{1}{8!}x^8 + \frac{1}{9!}x^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}x^{11} + \frac{1}{12!}x^{12} + \frac{1}{13!}x^{13} + \frac{1}{14!}x^{14} + \frac{1}{15!}x^{15} \\ & \frac{1}{16!}x^{16} + \frac{1}{17!}x^{17} + \frac{1}{18!}x^{18} + \frac{1}{19!}x^{19} + \frac{1}{20!}x^{20} + \frac{1}{21!}x^{21} + \frac{1}{22!}x^{22} + \frac{1}{23!}x^{23} \\ & \frac{1}{24!}x^{24} + \frac{1}{25!}x^{25} + \frac{1}{26!}x^{26} + \frac{1}{27!}x^{27} + \frac{1}{28!}x^{28} + \frac{1}{29!}x^{29} + \frac{1}{30!}x^{30} + \frac{1}{31!}x^{31} \\ & \frac{1}{32!}x^{32} + \frac{1}{33!}x^{33} + \frac{1}{34!}x^{34} + \frac{1}{35!}x^{35} + \frac{1}{36!}x^{36} + \frac{1}{37!}x^{37} + \frac{1}{38!}x^{38} + \frac{1}{39!}x^{39} \\ & + \cdots \end{aligned}$$

The outcome is

$$
\begin{aligned}
e^{ix} \;=\; & \Big(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12} - \frac{1}{14!}x^{14} \\
& + \frac{1}{16!}x^{16} - \frac{1}{18!}x^{18} + \frac{1}{20!}x^{20} - \frac{1}{22!}x^{22} + \frac{1}{24!}x^{24} - \frac{1}{26!}x^{26} + \frac{1}{28!}x^{28} - \frac{1}{30!}x^{30} \\
& + \frac{1}{32!}x^{32} - \frac{1}{34!}x^{34} + \frac{1}{36!}x^{36} - \frac{1}{38!}x^{38} + \cdots \Big) \\
& + i\Big(\frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \frac{1}{13!}x^{13} - \frac{1}{15!}x^{15} \\
& + \frac{1}{17!}x^{17} - \frac{1}{19!}x^{19} + \frac{1}{21!}x^{21} - \frac{1}{23!}x^{23} + \frac{1}{25!}x^{25} - \frac{1}{27!}x^{27} + \frac{1}{29!}x^{29} - \frac{1}{31!}x^{31} \\
& + \frac{1}{33!}x^{33} - \frac{1}{35!}x^{35} + \frac{1}{37!}x^{37} - \frac{1}{39!}x^{39} + \cdots \Big) .
\end{aligned}
$$

This is clearly

$$
\Big(\cos x\Big) + i\Big(\sin x\Big).
$$

This way we obtain

Euler's formula. $$\boxed{e^{ix} \;=\; \Big(\cos x\Big) \;+\; i\Big(\sin x\Big)}\;.$$

Now, this is not entirely absurd. On the contrary, this is actually very illuminating. Indeed, let

$$\alpha \;=\; e^{ix}, \qquad\qquad \beta \;=\; e^{iy}.$$

Then

$$\begin{aligned}\alpha\,\beta \;&=\; e^{ix}\,e^{iy}\\ &=\; \Bigg(\Big(\cos x\Big) \;+\; i\Big(\sin x\Big)\Bigg)\Bigg(\Big(\cos y\Big) \;+\; i\Big(\sin y\Big)\Bigg).\end{aligned}$$

Let's expand this, taking into account $i^2 \;=\; -1$:

$$(*) \qquad \alpha\,\beta \;=\; \Bigg[\Big(\cos x\Big)\Big(\cos y\Big) \;-\; \Big(\sin x\Big)\Big(\sin y\Big)\Bigg]$$

$$+\; i\;\Bigg[\Big(\cos x\Big)\Big(\sin y\Big) \;+\; \Big(\sin x\Big)\Big(\cos y\Big)\Bigg].$$

And that was just simply the multiplication. But then suddenly let's compare this with

Axiom 3. $$\boxed{\cos\Big(x+y\Big) \;=\; \Big(\cos x\Big)\Big(\cos y\Big) \;-\; \Big(\sin x\Big)\Big(\sin y\Big)}\;.$$

Axiom 4. $$\boxed{\sin\Big(x+y\Big) \;=\; \Big(\sin x\Big)\Big(\cos y\Big) \;+\; \Big(\cos x\Big)\Big(\sin y\Big)}\;.$$

You realize that the right-hand side of the identity $(*)$ on the past page is

$$\Big(\cos \big(x + y \big) \Big) \; + \; i \; \Big(\sin \big(x + y \big) \Big) .$$

But this is exactly $e^{i\,(x+y)}$. So, we have proved

Exponential Law – II. Let x and y be real numbers. Then

$$\boxed{ e^{ix} \, e^{iy} \;=\; e^{i\,(x+y)} } .$$

The original exponential law (‘Rule II’ in “Review of Lectures – XVIII”) is

Exponential Law (Original). Let x and y be real numbers. Then

$$\boxed{ e^{x} \, e^{y} \;=\; e^{x+y} } .$$

These two resemble each other. Actually, this is more than a resemblance. These two are considered to be a single formula branching off in two different directions. So we call both of these formulas ‘Exponential Law’. In that sense, in retrospect, ‘Axiom 3’ and ‘Axiom 4’ combined was a disguised form of the exponential law.