Math 105 TOPICS IN MATHEMATICS
STUDY GUIDE FOR FINAL EXAM – FB

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- §23. Polynomials and their arithmetic – II.

Q. Expand

(1) $x\left(x^2 - x + 4\right).$ (2) $x^3\left(2x^4 + 8x^3 + 5x\right).$

(3) $6x\left(-\dfrac{1}{3}x^3 + \dfrac{1}{2}x^2 - \dfrac{1}{6}x\right).$

(4) $\dfrac{1}{4}x^2\left(x^4 + 2x^3 + 3x^2 + 2x + 1\right).$

[**Answers**]:

(1) $x^3 - x^2 + 4x.$ (2) $2x^7 + 8x^6 + 5x^4.$

(3) $-2x^4 + 3x^3 - x^2.$ (4) $\dfrac{1}{4}x^6 + \dfrac{1}{2}x^5 + \dfrac{3}{4}x^4 + \dfrac{1}{2}x^3 + \dfrac{1}{4}x^2.$

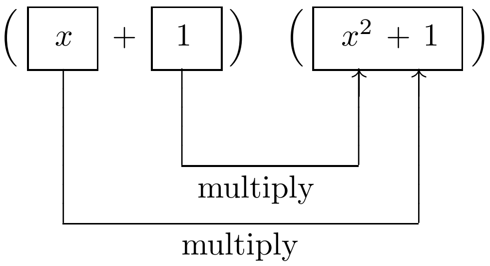
- **Multiplying out two polynomials.**

Even more generally, we may multiply out two polymials. The outcome is a polynomial. This process is once again called an expansion.

Example. Let's expand

$$\left(x + 1\right)\left(x^2 + 1\right).$$

It goes as follows.



$$= \ x \left(x^2 + 1\right) \ + \ 1 \left(x^2 + 1\right)$$

$$= \left(x^3 + x\right) \ + \ \left(x^2 + 1\right)$$

$$= \ x^3 + x + x^2 + 1$$

$$= \ x^3 + x^2 + x + 1.$$

In short,

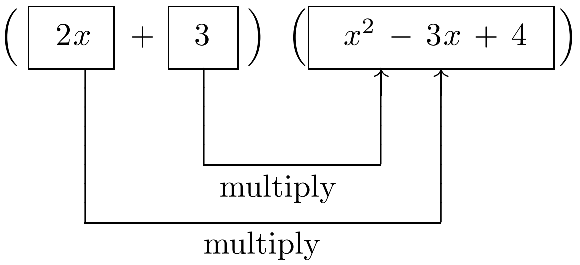
$$\left(x + 1\right)\left(x^2 + 1\right) \ = \ x^3 + x^2 + x + 1.$$

$\star$ As you can see, the above consists of two multiplications, both "monomial times polynomial" type, followed by one polynomial addition.

Example. Let's expand

$$\left(2x^2 + 3\right)\left(x^2 - 3x + 4\right).$$

It goes as follows.



$$= \ 2x \left(x^2 - 3x + 4\right) \quad + \quad 3 \left(x^2 - 3x + 4\right)$$

$$= \ \left(2x^3 - 6x^2 + 8x\right) \quad + \quad \left(3x^2 - 9x + 12\right)$$

$$= \ 2x^3 - 6x^2 + 8x + 3x^2 - 9x + 12$$

$$\left(\text{uncovered parentheses}\right)$$

$$= \ 2x^3 - 6x^2 + 3x^2 + 8x - 9x + 12$$

$$\left(\text{re-ordered terms}\right)$$

$$= \ 2x^3 - 3x^2 - x + 12.$$

In short,

$$\left(2x + 3\right)\left(x^2 - 3x + 4\right) \ = \ 2x^3 - 3x^2 - x + 12.$$

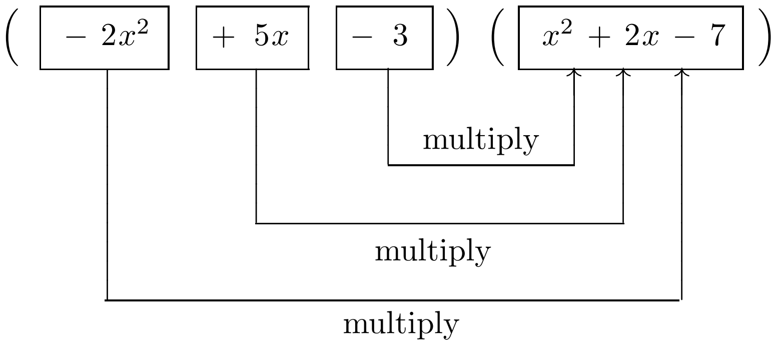
⋆ The following recaptures the same algorithm:

$$\begin{array}{rrrrrrrr}
 & & & x^2 & - & 3\,x & + & 4 \\
 & & & & & 2\,x & + & 3 \\
\times) & & & & & & & \\
\hline
 & & & 3\,x^2 & - & 9\,x & + & 12 \\
 & 2\,x^3 & - & 6\,x^2 & + & 8\,x & & \\
\hline
 & 2\,x^3 & - & 3\,x^2 & - & x & + & 12
\end{array}$$

Example. Let's expand

$$\Big(-\,2x^2\,+\,5x\,-\,3\Big)\Big(x^2\,+\,2x\,-\,7\Big).$$

It goes as follows.



$$= \;-2x^2\,\Big(x^2\,+\,2x\,-\,7\Big)\;\;+\;\;5x\,\Big(x^2\,+\,2x\,-\,7\Big)\;\;-\;\;3\,\Big(x^2\,+\,2x\,-\,7\Big)$$

$$= \;\Big(-\,2\,x^4-4x^3+14\,x^2\Big)\;\;+\;\;\Big(5\,x^3+10x^2-35\,x\Big)\;\;+\;\;\Big(-\,3x^2-6x+21\Big)$$

$$= \; -2x^4 - 4x^3 + 14x^2 + 5x^3 + 10x^2 - 35x - 3x^2 - 6x + 21$$

$$\Big(\text{uncovered parentheses}\Big)$$

$$= \; -2x^4 - 4x^3 + 5x^3 + 14x^2 + 10x^2 - 3x^2 - 35x - 6x + 21$$

$$\Big(\text{re-ordered terms}\Big)$$

$$= \; -2x^4 + x^3 + 21x^2 - 41x + 21.$$

In short,

$$\Big(-2x^2 + 5x - 3\Big)\Big(x^2 + 2x - 7\Big) \; = \; -2x^4 + x^3 + 21x^2 - 41x + 21.$$

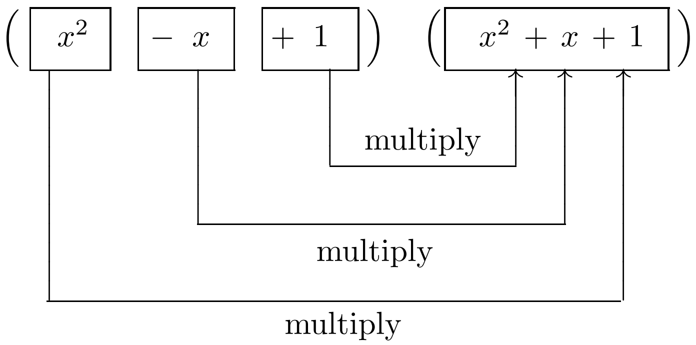
⋆ The following recaptures the same algorithm:

$$\begin{array}{rrrrrr}
 & & & x^2 & +\;2x & -\;7 \\
 & & & -\;2x^2 & +\;5x & -\;3 \\
\times) & & & & & \\
\hline
 & & & -\;3x^2 & -\;6x & +\;21 \\
 & & 5x^3 & +\;10x^2 & -\;35x & \\
 & -2x^4 & -\;4x^3 & +\;14x^2 & & \\
\hline
 & -2x^4 & +\;x^3 & +\;21x^2 & -\;41x & +\;21
\end{array}$$

Example. Let's expand

$$\Big(x^2 - x + 1\Big)\Big(x^2 + x + 1\Big).$$

It goes as follows.



$$= x^2 \left(x^2 + x + 1\right) \ - \ x \left(x^2 + x + 1\right) \ + \ 1 \left(x^2 + x + 1\right)$$

$$= \left(x^4 + x^3 + x^2\right) \ + \ \left(- x^3 - x^2 - x\right) \ + \ \left(x^2 + x + 1\right)$$

$$= x^4 + x^3 + x^2 - x^3 - x^2 - x + x^2 + x + 1$$

$$\left(\text{uncovered parentheses}\right)$$

$$= x^4 + x^3 - x^3 + x^2 - x^2 + x^2 - x + x + 1$$

$$\left(\text{re-ordered terms}\right)$$

$$= x^4 + x^2 + 1.$$

In short,

$$\left(x^2 - x + 1\right)\left(x^2 + x + 1\right) \ = \ x^4 + x^2 + 1.$$

⋆ The following is an alternative way:

$$
\begin{array}{rrrrrrrrrr}
 & & & & & x^2 & + & x & + & 1 \\
 & & & & & x^2 & - & x & + & 1 \\
\times) & & & & & & & & & \\
\hline
 & & & & & x^2 & + & x & + & 1 \\
 & & - & x^3 & - & x^2 & - & x & & \\
 & x^4 & + & x^3 & + & x^2 & & & & \\
\hline
 & x^4 & & & + & x^2 & & & + & 1
\end{array}
$$

- Let's do

(0) $\Big(x - 1\Big) \cdot 1,$

(1) $\Big(x - 1\Big)\Big(x + 1\Big),$

(2) $\Big(x - 1\Big)\Big(x^2 + x + 1\Big),$

(3) $\Big(x - 1\Big)\Big(x^3 + x^2 + x + 1\Big),$

(4) $\Big(x - 1\Big)\Big(x^4 + x^3 + x^2 + x + 1\Big),$

(5) $\Big(x - 1\Big)\Big(x^5 + x^4 + x^3 + x^2 + x + 1\Big),$

(6) $\Big(x - 1\Big)\Big(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\Big),$

$$\vdots$$

Let's try (5). Let's do it the second way:

$$\begin{array}{rccccccccccccc}
 & & & x^5 & + & x^4 & + & x^3 & + & x^2 & + & x & + & 1 \\
 & & & & & & & & & & & x & - & 1 \\
\times) & & & & & & & & & & & & & \\
\hline
 & & & -x^5 & - & x^4 & - & x^3 & - & x^2 & - & x & - & 1 \\
 & x^6 & + & x^5 & + & x^4 & + & x^3 & + & x^2 & + & x & & \\
\hline
 & x^6 & & & & & & & & & & & - & 1
\end{array}$$

In short,

(5) $$\Big(x - 1\Big)\Big(x^5 + x^4 + x^3 + x^2 + x + 1\Big) = x^6 - 1.$$

By extrapolation:

(0) $\Big(x - 1\Big) \cdot 1 = x - 1,$

(1) $\Big(x - 1\Big)\Big(x + 1\Big) = x^2 - 1,$

(2) $\Big(x - 1\Big)\Big(x^2 + x + 1\Big) = x^3 - 1,$

(3) $\Big(x - 1\Big)\Big(x^3 + x^2 + x + 1\Big) = x^4 - 1,$

(4) $\Big(x - 1\Big)\Big(x^4 + x^3 + x^2 + x + 1\Big) = x^5 - 1,$

(5) $\Big(x - 1\Big)\Big(x^5 + x^4 + x^3 + x^2 + x + 1\Big) = x^6 - 1,$

(6) $\Big(x - 1\Big)\Big(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\Big) = x^7 - 1,$

$$\vdots$$

⋆ A minor tweak:

(0) $\Big(1 - x\Big) \cdot 1 = 1 - x,$

(1) $\Big(1 - x\Big)\Big(1 + x\Big) = 1 - x^2,$

(2) $\Big(1 - x\Big)\Big(1 + x + x^2\Big) = 1 - x^3,$

(3) $\Big(1 - x\Big)\Big(1 + x + x^2 + x^3\Big) = 1 - x^4,$

(4) $\Big(1 - x\Big)\Big(1 + x + x^2 + x^3 + x^4\Big) = 1 - x^5,$

(5) $\Big(1 - x\Big)\Big(1 + x + x^2 + x^3 + x^4 + x^5\Big) = 1 - x^6,$

(6) $\Big(1 - x\Big)\Big(1 + x + x^2 + x^3 + x^4 + x^5 + x^6\Big) = 1 - x^7,$

$$\vdots$$

$\star$ Further tweak:

(0) $\Big(1 + x\Big) \cdot 1 = 1 + x,$

(1) $\Big(1 + x\Big)\Big(1 - x\Big) = 1 - x^2,$

(2) $\Big(1 + x\Big)\Big(1 - x + x^2\Big) = 1 + x^3,$

(3) $\Big(1 + x\Big)\Big(1 - x + x^2 - x^3\Big) = 1 - x^4,$

(4) $\Big(1 + x\Big)\Big(1 - x + x^2 - x^3 + x^4\Big) = 1 + x^5,$

(5) $\Big(1 + x\Big)\Big(1 - x + x^2 - x^3 + x^4 - x^5\Big) = 1 - x^6,$

(6) $\Big(1 + x\Big)\Big(1 - x + x^2 - x^3 + x^4 - x^5 + x^6\Big) = 1 + x^7,$

$\vdots$

Q. Expand

(1) $\Big(x + 3\Big)\Big(x + 4\Big).$ (2) $\Big(x + 5\Big)\Big(x - 2\Big).$

(3) $\Big(x^2 - 7\Big)\Big(x^2 - 3x + 1\Big).$ (4) $\Big(x^3 - 4x^2 + 2\Big)\Big(x^2 + 3x - 8\Big).$

$\Big[$ **Answers** $\Big]$**:**

(1) $x^2 + 7x + 12.$ (2) $x^2 + 3x - 10.$

(3) $x^4 - 3x^3 - 6x^2 + 21x - 7.$ (4) $x^5 - x^4 - 20x^3 + 34\,x^2 + 6x - 16.$

Q. Expand

(1) $$\Big(x - 1\Big)\Big(x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1\Big).$$

(2) $$\Big(x - 1\Big)\Big(x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1\Big).$$

(3) $$\Big(1 - x\Big)\Big(1 + x + x^{2} + x^{3} + x^{4} + x^{5} + x^{6} + x^{7} + x^{8} + x^{9} + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19} + x^{20} + x^{21} + x^{22} + x^{23}\Big).$$

(4) $$\Big(1 + x\Big)\Big(1 - x + x^{2} - x^{3} + x^{4} - x^{5} + x^{6} - x^{7} + x^{8} - x^{9} + x^{10} - x^{11} + x^{12} - x^{13} + x^{14}\Big).$$

(5) $$\Big(1 + x\Big)\Big(1 - x + x^{2} - x^{3} + x^{4} - x^{5} + x^{6} - x^{7} + x^{8} - x^{9} + x^{10} - x^{11} + x^{12} - x^{13} + x^{14} - x^{15} + x^{16} - x^{17} + x^{18} - x^{19} + x^{20} - x^{21} + x^{22} - x^{23} + x^{24} - x^{25} + x^{26} - x^{27} + x^{28} - x^{29} + x^{30} - x^{31} + x^{32} - x^{33} + x^{34} - x^{35} + x^{36} - x^{37} + x^{38} - x^{39} + x^{40} - x^{41} + x^{42} - x^{43} + x^{44} - x^{45} + x^{46} - x^{47}\Big).$$

[**Answers**]**:**

(1) $x^{21} - 1.$ (2) $x^{41} - 1.$

(3) $1 - x^{24}.$ (4) $1 + x^{15}.$

(5) $1 - x^{48}.$

- §24. Polynomials and their arithmetic – III.

Example. Let's expand

$$\left(x^2 + 3\right)^2.$$

This is the same as

$$\left(x^2 + 3\right)\left(x^2 + 3\right).$$

So

$$\begin{aligned}\left(x^2 + 3\right)\left(x^2 + 3\right) &= x^2\left(x^2 + 3\right) + 3\left(x^2 + 3\right) \\ &= x^4 + 3\,x^2 + 3\,x^2 + 9 \\ &= x^4 + 6\,x^2 + 9.\end{aligned}$$

This can be done using "binomial formula". Recall

$$\left(a + b\right)^2 = a^2 + 2ab + b^2.$$

So

$$\begin{aligned}\left(a + 3\right)^2 &= a^2 + 2 \cdot 3 \cdot a + 3^2 \\ &= a^2 + 6\,a + 9.\end{aligned}$$

Substitute a with x^2:

$$\begin{aligned}\left(x^2 + 3\right)^2 &= \left(x^2\right)^2 + 6\,x^2 + 9 \\ &= x^4 + 6\,x^2 + 9.\end{aligned}$$

We certainly get the same answer.

Example. Let's expand

$$\left(x^2 + x + 2\right)^2.$$

This time you have to do it 'honestly', like

$$\left(x^2 + x + 2\right)\left(x^2 + x + 2\right).$$

It goes as follows:

$$\begin{aligned}
&\left(x^2 + x + 2\right)\left(x^2 + x + 2\right) \\
&\qquad = x^2\left(x^2 + x + 2\right) + x\left(x^2 + x + 2\right) + 2\left(x^2 + x + 2\right) \\
&\qquad = \left(x^4 + x^3 + 2x^2\right) + \left(x^3 + x^2 + 2x\right) + \left(2x^2 + 2x + 4\right)
\end{aligned}$$

$$
\begin{aligned}
&= x^4 + x^3 + 2x^2 + x^3 + x^2 + 2x + 2x^2 + 2x + 4 \\
&= x^4 + x^3 + x^3 + 2x^2 + x^2 + 2x^2 + 2x + 2x + 4 \\
&= x^4 + 2x^3 + 5x^2 + 4x + 4.
\end{aligned}
$$

You can certainly do it this way:

$$
\begin{array}{r|ccccccccc}
 & & & & & x^2 & + & x & + & 2 \\
 & & & & & x^2 & + & x & + & 2 \\
\times) & & & & & & & & & \\
\hline
 & & & & & 2\,x^2 & + & 2\,x & + & 4 \\
 & & & x^3 & + & x^2 & + & 2\,x & & \\
 & x^4 & + & x^3 & + & 2\,x^2 & & & & \\
\hline
 & x^4 & + & 2\,x^3 & + & 5\,x^2 & + & 4\,x & + & 4
\end{array}
$$

To conclude,

$$\left(x^2 + x + 2\right)^2 = x^4 + 2x^3 + 5x^2 + 4x + 4.$$

Example. Let's expand

$$\left(x^3 + 4x^2 - 3x + 2\right)^2.$$

Let's just do it in the second way.

$$
\begin{array}{lrrrrrrr}
 & & & & x^3 & +\ 4\,x^2 & -\ 3\,x & +\ 2 \\
\times) & & & & x^3 & +\ 4\,x^2 & -\ 3\,x & +\ 2 \\
\hline
 & & & & 2\,x^3 & +\ 8\,x^2 & -\ 6\,x & +\ 4 \\
 & & & -\ 3\,x^4 & -\ 12\,x^3 & +\ 9\,x^2 & -\ 6\,x & \\
 & & 4\,x^5 & +\ 16\,x^4 & -\ 12\,x^3 & +\ 8\,x^2 & & \\
 & x^6 & +\ 4\,x^5 & -\ 3\,x^4 & +\ 2\,x^3 & & & \\
\hline
 & x^6 & +\ 8\,x^5 & +\ 10\,x^4 & -\ 20\,x^3 & +\ 25\,x^2 & -\ 12\,x & +\ 4
\end{array}
$$

To conclude,

$$\left(x^3 + 4x^2 - 3x + 2\right)^2 = x^6 + 8x^5 + 10x^4 - 20x^3 + 25x^2 - 12x + 4.$$

Example. Let's expand $\left(x^4 - 5x^3 - 6\,x + 4\right)^2$.

As before, we can handle it like

$$
\begin{array}{lrrrrrrrrr}
 & & & & & x^4 & -\ 5\,x^3 & & -\ 6\,x & +\ 4 \\
\times) & & & & & x^4 & -\ 5\,x^3 & & -\ 6\,x & +\ 4 \\
\hline
 & & & & & 4\,x^4 & -\ 20\,x^3 & & -\ 24\,x & +\ 16 \\
 & & & & -\ 6\,x^5 & +\ 30\,x^4 & & +\ 36\,x^2 & -\ 24\,x & \\
 & & -\ 5\,x^7 & +\ 25\,x^6 & & +\ 30\,x^4 & -\ 20\,x^3 & & & \\
 & x^8 & -\ 5\,x^7 & & -\ 6\,x^5 & +\ 4\,x^4 & & & & \\
\hline
 & x^8 & -\ 10\,x^7 & +\ 25\,x^6 & -\ 12\,x^5 & +\ 68\,x^4 & -\ 40\,x^3 & +\ 36\,x^2 & -\ 48\,x & +\ 16
\end{array}
$$

To conclude,

$$\left(x^4 - 5x^3 - 6\,x + 4\right)^2$$

$$= \; x^8 - 10x^7 + 25x^6 - 12x^5 + 68x^4 - 40x^3 + 36x^2 - 48x + 16.$$

Example. Let's expand $\left(x^4 + x^3 + x^2 + x + 1\right)^2$.

The same deal:

$$
\begin{array}{rrrrrrrrrr}
 & & & & & x^4 & + \; x^3 & + \; x^2 & + \; x & + \; 1 \\
\times) & & & & & x^4 & + \; x^3 & + \; x^2 & + \; x & + \; 1 \\
\hline
 & & & & & x^4 & + \; x^3 & + \; x^2 & + \; x & + \; 1 \\
 & & & & x^5 & + \; x^4 & + \; x^3 & + \; x^2 & + \; x & \\
 & & & x^6 & + \; x^5 & + \; x^4 & + \; x^3 & + \; x^2 & & \\
 & & x^7 & + \; x^6 & + \; x^5 & + \; x^4 & + \; x^3 & & & \\
 & x^8 & + \; x^7 & + \; x^6 & + \; x^5 & + \; x^4 & & & & \\
\hline
 & x^8 & + 2x^7 & + 3x^6 & + 4x^5 & + 5x^4 & + 4x^3 & + 3x^2 & + 2x & + \; 1
\end{array}
$$

To conclude,

$$\left(x^4 + x^3 + x^2 + x + 1\right)^2$$

$$= \; x^8 + 2x^7 + 3x^6 + 4x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1.$$

Q. Expand

(1) $\left(x^3 - 8x\right)^2$. (2) $\left(2x^2 - 3x + 5\right)^2$.

(3) $\left(x^3 + x^2 - 4\right)^2$. (4) $\left(x^2 + \frac{1}{2}x + \frac{1}{3}\right)^2$.

(5) $\left(1 + x + x^2 + x^3 + x^4 + x^5\right)^2$.

$\left[\underline{\textbf{Answers}}\right]$**:**

(1) $x^6 - 16x^4 + 64x^2$. (2) $4x^4 - 12x^3 + 29x^2 - 30x + 25$.

(3) $x^6 + 2x^5 + x^4 - 8x^3 - 8x^2 + 16$.

(4) $x^4 + x^3 + \frac{11}{12}x^2 + \frac{11}{3}x + \frac{11}{9}$.

(5) $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 5x^6 + 4x^7 + 3x^8 + 2x^9 + x^{10}$.

- **Product of three or more polynomials.**

Example. How about

$$\left(x + 1\right)\left(x^2 + x + 1\right)\left(x^4 + x^2 + 1\right)?$$

There are three polynomials involved. You have to do it step by step. In

$$\boxed{\left(x + 1\right)\left(x^2 + x + 1\right)}\ \left(x^4 + x^2 + 1\right)$$

you first do the boxed part, and then multiply the outcome with the third factor $x^4 + x^2 + 1$.

Step 1. Do $\Big(x + 1\Big)\Big(x^2 + x + 1\Big)$:

$$\begin{array}{rrrrrrrr}
 & & & x^2 & + & x & + & 1 \\
 & & & & & x & + & 1 \\
\times) & & & & & & & \\ \hline
 & & & x^2 & + & x & + & 1 \\
 & x^3 & + & x^2 & + & x & & \\ \hline
 & x^3 & + & 2\,x^2 & + & 2\,x & + & 1
\end{array}$$

In short,

$$\Big(x + 1\Big)\Big(x^2 + x + 1\Big) = x^3 + 2x^2 + 2x + 1.$$

Step 2. Do $\Big(x^3 + 2x^2 + 2x + 1\Big)\Big(x^4 + x^2 + 1\Big)$:

$$\begin{array}{rrrrrrrrrrrrrrrr}
 & & & & & & & x^4 & & & + & x^2 & & & + & 1 \\
 & & & & & & & & & x^3 & + & 2\,x^2 & + & 2\,x & + & 1 \\
\times) & & & & & & & & & & & & & & & \\ \hline
 & & & & & & & x^4 & & & + & x^2 & & & + & 1 \\
 & & & & & 2\,x^5 & & & + & 2\,x^3 & & & + & 2\,x & & \\
 & & & 2\,x^6 & & & + & 2\,x^4 & & & + & 2\,x^2 & & & & \\
 & x^7 & & & + & x^5 & & & + & x^3 & & & & & & \\ \hline
 & x^7 & + & 2\,x^6 & + & 3\,x^5 & + & 3\,x^4 & + & 3\,x^3 & + & 3\,x^2 & + & 2\,x & + & 1
\end{array}$$

In short,

$$\begin{aligned}
&\Big(x^3 + 2x^2 + 2x + 1\Big)\Big(x^4 + x^2 + 1\Big) \\
&\qquad = x^7 + 2x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 2x + 1.
\end{aligned}$$

To conclude,

$$\begin{aligned}&\Big(x + 1\Big)\Big(x^2 + x + 1\Big)\Big(x^4 + x^2 + 1\Big)\\ &\quad = x^7 + 2x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 2x + 1.\end{aligned}$$

Q. Expand:

(1) $\Big(x - 1\Big)\Big(x + 1\Big)^2.$

(2) $\Big(x - 1\Big)\Big(x - 3\Big)\Big(x^2 - 3\Big).$

(3) $\Big(x - 1\Big)\Big(x + 1\Big)\Big(x^2 + 1\Big).$

(4) $\Big(x^2 - \sqrt{2}\,x + 1\Big)\Big(x^2 + \sqrt{2}\,x + 1\Big)\Big(x^2 - 1\Big).$

(5) $\Big(x - \sqrt{2} - 1\Big)\Big(x + \sqrt{2} - 1\Big)\Big(x - \sqrt{2} + 1\Big)\Big(x + \sqrt{2} + 1\Big).$

[**Answers**]**:**

(1) $x^3 + x^2 - x - 1.$

(2) $x^4 - 4x^3 + 12x - 9.$

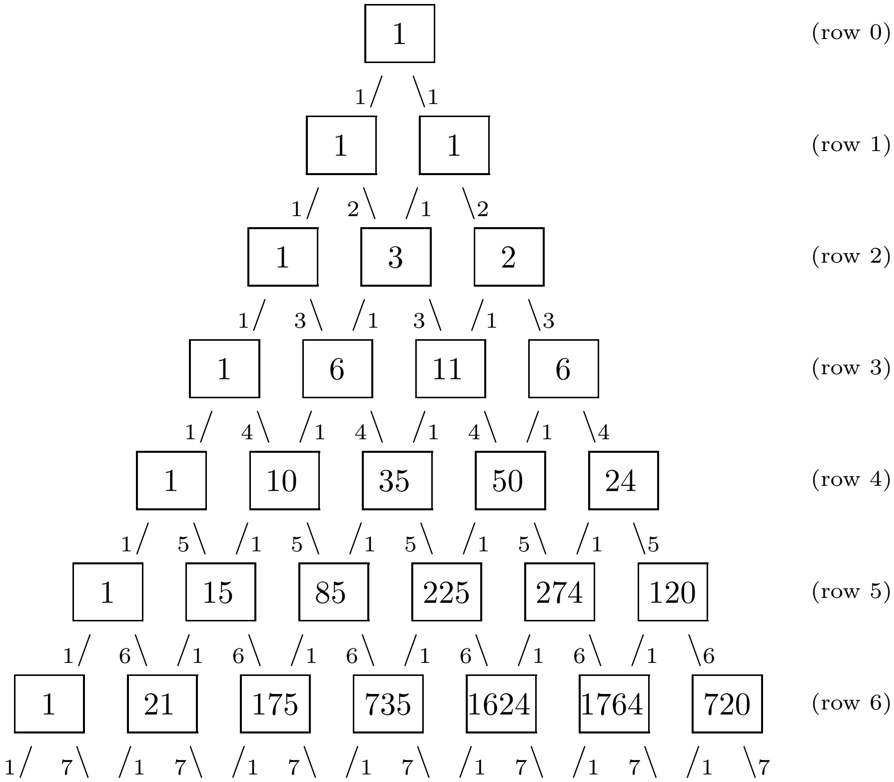
(3) $x^4 - 1.$

(4) $x^6 - x^4 + x^2 - 1.$

(5) $x^4 - 6\,x^2 + 1.$

- **Raising products.**

The following has some bearings on certain types of polynomial multiplications:



The above — apparently a variation of Pascal — can be used to get the expansions

$$
\begin{aligned}
&\Big(x+1\Big),\\
&\Big(x+1\Big)\Big(x+2\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big)\Big(x+5\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big)\Big(x+5\Big)\Big(x+6\Big),\\
&\qquad\vdots\qquad\qquad\qquad\qquad\qquad\qquad\ddots
\end{aligned}
$$

Namely:

$$\Big(x + 1\Big)\Big(x + 2\Big) \;=\; x^2 \;+\; 3\,x \;+\; 2,$$

$$\Big(x + 1\Big)\Big(x + 2\Big)\Big(x + 3\Big) \;=\; x^3 \;+\; 6\,x^2 \;+\; 11\,x \;+\; 6,$$

$$\Big(x + 1\Big)\Big(x + 2\Big)\Big(x + 3\Big)\Big(x + 4\Big)$$
$$=\; x^4 \;+\; 10\,x^3 \;+\; 35\,x^2 \;+\; 50\,x \;+\; 24,$$

$$\Big(x + 1\Big)\Big(x + 2\Big)\Big(x + 3\Big)\Big(x + 4\Big)\Big(x + 5\Big)$$
$$=\; x^5 \;+\; 15\,x^4 \;+\; 85\,x^3 \;+\; 225\,x^2 \;+\; 274\,x \;+\; 120,$$

$$\Big(x + 1\Big)\Big(x + 2\Big)\Big(x + 3\Big)\Big(x + 4\Big)\Big(x + 5\Big)\Big(x + 6\Big)$$
$$=\; x^6 \;+\; 21\,x^5 \;+\; 175\,x^4 \;+\; 735\,x^3 \;+\; 1624\,x^2 \;+\; 1764\,x \;+\; 720.$$

Q. Expand:

$$\Big(x + 1\Big)\Big(x + 2\Big)\Big(x + 3\Big)\Big(x + 4\Big)\Big(x + 5\Big)\Big(x + 6\Big)\Big(x + 7\Big).$$

⋆ As for this, the triangle in the previous page is apparently shown only up to the sixth row. You have to extend it to the seventh row.

[**Answer**]:

$$x^7 + 28x^6 + 322x^5 + 1960x^4 + 6769x^3 + 13132x^2 + 13068x + 5040.$$

- §25. Derivatives of polynomials.

First rule.

$$\begin{aligned} \frac{d}{dx}\,1 &= 0, \\ \frac{d}{dx}\,x &= 1, \\ \frac{d}{dx}\,x^2 &= 2x, \\ \frac{d}{dx}\,x^3 &= 3x^2, \\ \frac{d}{dx}\,x^4 &= 4x^3, \\ \frac{d}{dx}\,x^5 &= 5x^4, \\ \frac{d}{dx}\,x^6 &= 6x^5, \\ &\vdots \end{aligned}$$

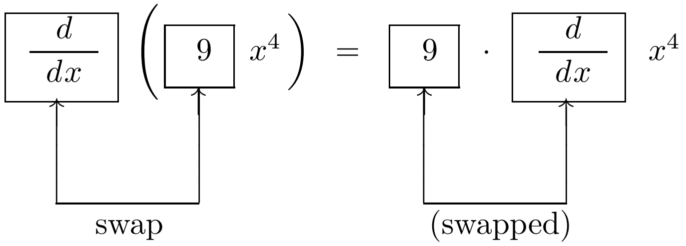
More generally:

Definition. Let n be an integer, $n \geq 0$. Then

$$\boxed{\frac{d}{dx}\,x^n = n\,x^{n-1}}\,.$$

Next, how you handle constant multiplication inside $\dfrac{d}{dx}$.

Example. $\dfrac{d}{dx}\Big(9x^4\Big).$ This goes as follows:

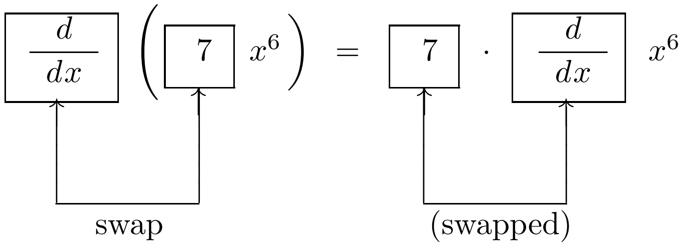


$$\boxed{\frac{d}{dx}}\left(\boxed{9}\;x^4\right) = \boxed{9}\cdot\boxed{\frac{d}{dx}}\;x^4$$

$$= \quad 9 \cdot 4 \cdot x^3$$

$$= \quad 36 \cdot x^3.$$

Example 2. $\dfrac{d}{dx}\Big(7x^6\Big).$ This goes as follows:

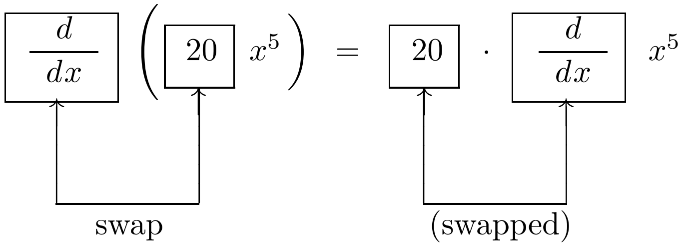


$$\boxed{\frac{d}{dx}}\left(\boxed{7}\;x^6\right) = \boxed{7}\cdot\boxed{\frac{d}{dx}}\;x^6$$

$$= \quad 7 \cdot 6 \cdot x^5$$

$$= \quad 42 \cdot x^5.$$

Example 3. $\frac{d}{dx}\Big(20x^5\Big).$ This goes as follows:



$$\boxed{\frac{d}{dx}}\left(\boxed{20}\;x^5\right) \;=\; \boxed{20}\cdot\boxed{\frac{d}{dx}}\;x^5$$

$$= \; 20 \,\cdot\, 5 \,\cdot\, x^4$$

$$= \; 100 \,\cdot\, x^4.$$

⋆ In short, the rule is, whenever you see a constant inside $\frac{d}{dx}$, drag it outside $\frac{d}{dx}$. You may write the above three results as

$$\frac{d}{dx}\Big(9x^4\Big) \;=\; 9\cdot 4\cdot x^3 \;=\; 36\cdot x^3,$$

$$\frac{d}{dx}\Big(14x^6\Big) \;=\; 14\cdot 6\cdot x^5 \;=\; 84\cdot x^5,$$

$$\frac{d}{dx}\Big(20x^5\Big) \;=\; 20\cdot 5\cdot x^4 \;=\; 100\cdot x^4.$$

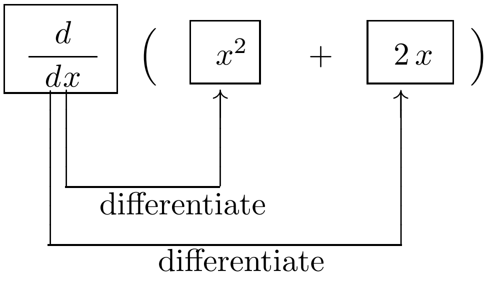
and so on. The following definition makes it official:

Definition. Let a be a constant real number. Let n be an integer, $n \geq 0$. Then

$$\boxed{\frac{d}{dx}\Big(ax^n\Big) \;=\; a\,\frac{d}{dx}\,x^n \;=\; a\,n\,x^{n-1}}\;.$$

⋆ Finally, $\dfrac{d}{dx}$ breaks up addition.

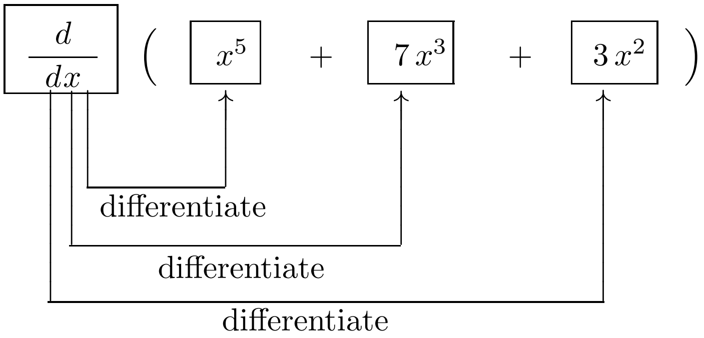
Example 4. $\dfrac{d}{dx}\Big(x^2 + 2x\Big)$. This goes as follows:



$$= \left[\frac{d}{dx}\,x^2\right] + \left[\frac{d}{dx}\Big(2x\Big)\right]$$

$$= 2x + 2.$$

Example 5. $\dfrac{d}{dx}\Big(x^5 + 7x^3 + 3x^2\Big)$. This goes as follows:



$$= \left[\frac{d}{dx}\,x^5\right] + \left[\frac{d}{dx}\Big(7x^3\Big)\right] + \left[\frac{d}{dx}\Big(3x^2\Big)\right]$$

$$= 5x^4 + 7\cdot 3x^2 + 3\cdot 2x = 5x^4 + 21x^2 + 6x.$$

The following definition makes it official:

Definition. Let $f(x)$ and $g(x)$ be monomials. Then

$$\boxed{\frac{d}{dx}\Big(f(x) + g(x)\Big) = f'(x) + g'(x)}.$$

$$\boxed{\frac{d}{dx}\Big(f(x) + g(x) + h(x)\Big) = f'(x) + g'(x) + h'(x)}.$$

⋆ As for four or more monomials, the same thing.

Example. Let's differentiate

$$f(x) = x^2 + 2x + 5.$$

It goes as follows:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\Big(x^2 + 2x + 5\Big) \\ &= \left[\frac{d}{dx}x^2\right] + \left[\frac{d}{dx}\Big(2x\Big)\right] + \left[\frac{d}{dx}5\right] \\ &= 2x + 2 + 0 \\ &= 2x + 2. \end{aligned}$$

Example. Let's differentiate

$$f\left(x\right) \;=\; 4x^4 \,+\, 8x^3 \,-\, 7x^2.$$

It goes as follows:

$$\begin{aligned} f'\left(x\right) \;&=\; \frac{d}{dx}\left(4x^4 \,+\, 8x^3 \,-\, 7x^2\right) \\ &=\; \left[\frac{d}{dx}\left(4x^4\right)\right] \,+\, \left[\frac{d}{dx}\left(8x^3\right)\right] \,-\, \left[\frac{d}{dx}\left(7x^2\right)\right] \\ &=\; 4\cdot 4x^3 \,+\, 8\cdot 3x^2 \,-\, 7\cdot 2x \\ &=\; 16x^3 \,+\, 24x^2 \,-\, 14x. \end{aligned}$$

Q. Do the following differentiation:

(1) $\dfrac{d}{dx}\,x^{10}.$ (2) $\dfrac{d}{dx}\,2x^8.$ (3) $\dfrac{d}{dx}\,4x^{50}.$

(4) $\dfrac{d}{dx}\,x^{1000}.$

$\left[\,\underline{\textbf{Answers}}\,\right]$**:**

(1) $10x^9.$ (2) $16x^7.$

(3) $200x^{49}.$ (4) $1000x^{999}.$

Q. Do the following differentiation:

(1) $$\frac{d}{dx}\left(2x^3 - 11x^2 + 4x\right).$$

(2) $$\frac{d}{dx}\left(8x^8 - 12x^6 + 24x^4 - 64x^2 + 96\right).$$

(3) $$\frac{d}{dx}\left(\frac{1}{4}x^4 - \frac{1}{2}x^3 + \frac{1}{6}x^2 - \frac{1}{30}\right).$$

(4) $$\frac{d}{dx}\left(\frac{10}{7}x^7 + \frac{3}{2}x^6 + 3x^5\right).$$

[**Answers**]**:**

(1) $6x^2 - 22x + 4.$ (2) $64x^7 - 72x^5 + 96x^3 - 128x.$

(3) $x^3 - \frac{3}{2}x^2 + \frac{1}{3}x.$ (4) $10x^6 + 9x^5 + 15x^4.$

Q. Do the following differentiation:

(1) $$\frac{d}{dx}\left(\frac{1}{1!}\,x\right).$$

(2) $$\frac{d}{dx}\left(\frac{1}{2!}\,x^2\right).$$

(3) $$\frac{d}{dx}\left(\frac{1}{3!}\,x^3\right).$$

(4) $$\frac{d}{dx}\left(\frac{1}{4!}\,x^4\right).$$

(5) $$\frac{d}{dx}\left(\frac{1}{5!}\,x^5\right).$$

(5) $$\frac{d}{dx}\left(\frac{1}{6!}\,x^6\right).$$

$\left[\underline{\textbf{ Answers }}\right]$**:**

(1) $$\frac{d}{dx}\left(\frac{1}{1!}\,x\right) \quad = \quad 1.$$

(2) $$\frac{d}{dx}\left(\frac{1}{2!}\,x^2\right) \quad = \quad \frac{1}{1!}\,x \quad \Big(= \quad x\Big).$$

(3) $$\frac{d}{dx}\left(\frac{1}{3!}\,x^3\right) \quad = \quad \frac{1}{2!}\,x^2.$$

(4) $$\frac{d}{dx}\left(\frac{1}{4!}\,x^4\right) \quad = \quad \frac{1}{3!}\,x^3.$$

(5) $$\frac{d}{dx}\left(\frac{1}{5!}\,x^5\right) \quad = \quad \frac{1}{4!}\,x^4.$$

(6) $$\frac{d}{dx}\left(\frac{1}{6!}\,x^6\right) \quad = \quad \frac{1}{5!}\,x^5.$$

Pop Quiz. $$\frac{d}{dx}\left(\frac{1}{20!}\,x^{20}\right) \quad =?$$

$\left[\underline{\textbf{ Answer }}\right]$**:** $$\frac{1}{19!}\,x^{19}.$$

Pop Quiz. $$\frac{d}{dx}\left(\frac{1}{100!}\,x^{100}\right) \quad =?$$

$\left[\underline{\textbf{ Answer }}\right]$**:** $$\frac{1}{99!}\,x^{99}.$$

Pop Quiz. Do the following differentiation:

$$\frac{d}{dx}\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8\right).$$

$\left[\underline{\textbf{Answer}}\right]$**:** $1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7.$

Pop Quiz. Do the following differentiation:

$$\begin{aligned}\frac{d}{dx}\Bigg(&1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8+\frac{1}{9!}x^9\\
&+\frac{1}{10!}x^{10}+\frac{1}{11!}x^{11}+\frac{1}{12!}x^{12}+\frac{1}{13!}x^{13}+\frac{1}{14!}x^{14}+\frac{1}{15!}x^{15}+\frac{1}{16!}x^{16}+\frac{1}{17!}x^{17}+\frac{1}{18!}x^{18}+\frac{1}{19!}x^{19}\\
&+\frac{1}{20!}x^{20}+\frac{1}{21!}x^{21}+\frac{1}{22!}x^{22}+\frac{1}{23!}x^{23}+\frac{1}{24!}x^{24}+\frac{1}{25!}x^{25}+\frac{1}{26!}x^{26}+\frac{1}{27!}x^{27}+\frac{1}{28!}x^{28}+\frac{1}{29!}x^{29}\\
&+\frac{1}{30!}x^{30}+\frac{1}{31!}x^{31}+\frac{1}{32!}x^{32}+\frac{1}{33!}x^{33}+\frac{1}{34!}x^{34}+\frac{1}{35!}x^{35}+\frac{1}{36!}x^{36}+\frac{1}{37!}x^{37}+\frac{1}{38!}x^{38}+\frac{1}{39!}x^{39}\\
&+\frac{1}{40!}x^{40}+\frac{1}{41!}x^{41}+\frac{1}{42!}x^{42}+\frac{1}{43!}x^{43}+\frac{1}{44!}x^{44}+\frac{1}{45!}x^{45}+\frac{1}{46!}x^{46}+\frac{1}{47!}x^{47}+\frac{1}{48!}x^{48}+\frac{1}{49!}x^{49}\Bigg).\end{aligned}$$

$\left[\underline{\textbf{Answer}}\right]$**:**

$$\begin{aligned}&1\;+\;\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8+\frac{1}{9!}x^9\\
&+\frac{1}{10!}x^{10}+\frac{1}{11!}x^{11}+\frac{1}{12!}x^{12}+\frac{1}{13!}x^{13}+\frac{1}{14!}x^{14}+\frac{1}{15!}x^{15}+\frac{1}{16!}x^{16}+\frac{1}{17!}x^{17}+\frac{1}{18!}x^{18}+\frac{1}{19!}x^{19}\\
&+\frac{1}{20!}x^{20}+\frac{1}{21!}x^{21}+\frac{1}{22!}x^{22}+\frac{1}{23!}x^{23}+\frac{1}{24!}x^{24}+\frac{1}{25!}x^{25}+\frac{1}{26!}x^{26}+\frac{1}{27!}x^{27}+\frac{1}{28!}x^{28}+\frac{1}{29!}x^{29}\\
&+\frac{1}{30!}x^{30}+\frac{1}{31!}x^{31}+\frac{1}{32!}x^{32}+\frac{1}{33!}x^{33}+\frac{1}{34!}x^{34}+\frac{1}{35!}x^{35}+\frac{1}{36!}x^{36}+\frac{1}{37!}x^{37}+\frac{1}{38!}x^{38}+\frac{1}{39!}x^{39}\\
&+\frac{1}{40!}x^{40}+\frac{1}{41!}x^{41}+\frac{1}{42!}x^{42}+\frac{1}{43!}x^{43}+\frac{1}{44!}x^{44}+\frac{1}{45!}x^{45}+\frac{1}{46!}x^{46}+\frac{1}{47!}x^{47}+\frac{1}{48!}x^{48}.\end{aligned}$$

Pop Quiz. Do the following differentiation:

$$\frac{d}{dx}\Bigg(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8+\frac{1}{9!}x^9$$

$$+\frac{1}{10!}x^{10}+\frac{1}{11!}x^{11}+\frac{1}{12!}x^{12}+\frac{1}{13!}x^{13}+\frac{1}{14!}x^{14}+\frac{1}{15!}x^{15}+\frac{1}{16!}x^{16}+\frac{1}{17!}x^{17}+\frac{1}{18!}x^{18}+\frac{1}{19!}x^{19}$$

$$+\frac{1}{20!}x^{20}+\frac{1}{21!}x^{21}+\frac{1}{22!}x^{22}+\frac{1}{23!}x^{23}+\frac{1}{24!}x^{24}+\frac{1}{25!}x^{25}+\frac{1}{26!}x^{26}+\frac{1}{27!}x^{27}+\frac{1}{28!}x^{28}+\frac{1}{29!}x^{29}$$

$$+\frac{1}{30!}x^{30}+\frac{1}{31!}x^{31}+\frac{1}{32!}x^{32}+\frac{1}{33!}x^{33}+\frac{1}{34!}x^{34}+\frac{1}{35!}x^{35}+\frac{1}{36!}x^{36}+\frac{1}{37!}x^{37}+\frac{1}{38!}x^{38}+\frac{1}{39!}x^{39}$$

$$+\frac{1}{40!}x^{40}+\frac{1}{41!}x^{41}+\frac{1}{42!}x^{42}+\frac{1}{43!}x^{43}+\frac{1}{44!}x^{44}+\frac{1}{45!}x^{45}+\frac{1}{46!}x^{46}+\frac{1}{47!}x^{47}+\frac{1}{48!}x^{48}+\frac{1}{49!}x^{49}$$

$$+\frac{1}{50!}x^{50}+\frac{1}{51!}x^{51}+\frac{1}{52!}x^{52}+\frac{1}{53!}x^{53}+\frac{1}{54!}x^{54}+\frac{1}{55!}x^{55}+\frac{1}{56!}x^{56}+\frac{1}{57!}x^{57}+\frac{1}{58!}x^{58}+\frac{1}{59!}x^{59}$$

$$+\frac{1}{60!}x^{60}+\frac{1}{61!}x^{61}+\frac{1}{62!}x^{62}+\frac{1}{63!}x^{63}+\frac{1}{64!}x^{64}+\frac{1}{65!}x^{65}+\frac{1}{66!}x^{66}+\frac{1}{67!}x^{67}+\frac{1}{68!}x^{68}+\frac{1}{69!}x^{69}$$

$$+\frac{1}{70!}x^{70}+\frac{1}{71!}x^{71}+\frac{1}{72!}x^{72}+\frac{1}{73!}x^{73}+\frac{1}{74!}x^{74}+\frac{1}{75!}x^{75}+\frac{1}{76!}x^{76}+\frac{1}{77!}x^{77}+\frac{1}{78!}x^{78}+\frac{1}{79!}x^{79}$$

$$+\frac{1}{80!}x^{80}+\frac{1}{81!}x^{81}+\frac{1}{82!}x^{82}+\frac{1}{83!}x^{83}+\frac{1}{84!}x^{84}+\frac{1}{85!}x^{85}+\frac{1}{86!}x^{86}+\frac{1}{87!}x^{87}+\frac{1}{88!}x^{88}+\frac{1}{89!}x^{89}$$

$$+\frac{1}{90!}x^{90}+\frac{1}{91!}x^{91}+\frac{1}{92!}x^{92}+\frac{1}{93!}x^{93}+\frac{1}{94!}x^{94}+\frac{1}{95!}x^{95}+\frac{1}{96!}x^{96}+\frac{1}{97!}x^{97}+\frac{1}{98!}x^{98}+\frac{1}{99!}x^{99}$$

$$+\frac{1}{100!}x^{100}\Bigg).$$

$\left[\underline{\textbf{Answer}}\right]$:

$$
\begin{aligned}
&1 \;+\; \frac{1}{1!}x+\frac{1}{2!}x^{2}+\frac{1}{3!}x^{3}+\frac{1}{4!}x^{4}+\frac{1}{5!}x^{5}+\frac{1}{6!}x^{6}+\frac{1}{7!}x^{7}+\frac{1}{8!}x^{8}+\frac{1}{9!}x^{9}\\
&+\frac{1}{10!}x^{10}+\frac{1}{11!}x^{11}+\frac{1}{12!}x^{12}+\frac{1}{13!}x^{13}+\frac{1}{14!}x^{14}+\frac{1}{15!}x^{15}+\frac{1}{16!}x^{16}+\frac{1}{17!}x^{17}+\frac{1}{18!}x^{18}+\frac{1}{19!}x^{19}\\
&+\frac{1}{20!}x^{20}+\frac{1}{21!}x^{21}+\frac{1}{22!}x^{22}+\frac{1}{23!}x^{23}+\frac{1}{24!}x^{24}+\frac{1}{25!}x^{25}+\frac{1}{26!}x^{26}+\frac{1}{27!}x^{27}+\frac{1}{28!}x^{28}+\frac{1}{29!}x^{29}\\
&+\frac{1}{30!}x^{30}+\frac{1}{31!}x^{31}+\frac{1}{32!}x^{32}+\frac{1}{33!}x^{33}+\frac{1}{34!}x^{34}+\frac{1}{35!}x^{35}+\frac{1}{36!}x^{36}+\frac{1}{37!}x^{37}+\frac{1}{38!}x^{38}+\frac{1}{39!}x^{39}\\
&+\frac{1}{40!}x^{40}+\frac{1}{41!}x^{41}+\frac{1}{42!}x^{42}+\frac{1}{43!}x^{43}+\frac{1}{44!}x^{44}+\frac{1}{45!}x^{45}+\frac{1}{46!}x^{46}+\frac{1}{47!}x^{47}+\frac{1}{48!}x^{48}+\frac{1}{49!}x^{49}\\
&+\frac{1}{50!}x^{50}+\frac{1}{51!}x^{51}+\frac{1}{52!}x^{52}+\frac{1}{53!}x^{53}+\frac{1}{54!}x^{54}+\frac{1}{55!}x^{55}+\frac{1}{56!}x^{56}+\frac{1}{57!}x^{57}+\frac{1}{58!}x^{58}+\frac{1}{59!}x^{59}\\
&+\frac{1}{60!}x^{60}+\frac{1}{61!}x^{61}+\frac{1}{62!}x^{62}+\frac{1}{63!}x^{63}+\frac{1}{64!}x^{64}+\frac{1}{65!}x^{65}+\frac{1}{66!}x^{66}+\frac{1}{67!}x^{67}+\frac{1}{68!}x^{68}+\frac{1}{69!}x^{69}\\
&+\frac{1}{70!}x^{70}+\frac{1}{71!}x^{71}+\frac{1}{72!}x^{72}+\frac{1}{73!}x^{73}+\frac{1}{74!}x^{74}+\frac{1}{75!}x^{75}+\frac{1}{76!}x^{76}+\frac{1}{77!}x^{77}+\frac{1}{78!}x^{78}+\frac{1}{79!}x^{79}\\
&+\frac{1}{80!}x^{80}+\frac{1}{81!}x^{81}+\frac{1}{82!}x^{82}+\frac{1}{83!}x^{83}+\frac{1}{84!}x^{84}+\frac{1}{85!}x^{85}+\frac{1}{86!}x^{86}+\frac{1}{87!}x^{87}+\frac{1}{88!}x^{88}+\frac{1}{89!}x^{89}\\
&+\frac{1}{90!}x^{90}+\frac{1}{91!}x^{91}+\frac{1}{92!}x^{92}+\frac{1}{93!}x^{93}+\frac{1}{94!}x^{94}+\frac{1}{95!}x^{95}+\frac{1}{96!}x^{96}+\frac{1}{97!}x^{97}+\frac{1}{98!}x^{98}+\frac{1}{99!}x^{99}.
\end{aligned}
$$

- §28. Antiderivatives.

We all know that $2\,x$ is the derivative of x^2. Then we say x^2 is *an* antiderivative of $2\,x$. More generally:

Definition (Antiderivatives).

If a polynomial $\boxed{f(x)}$ is the derivative of another polynomial $\boxed{F(x)}$,

then we call $\boxed{F(x)}$ *an* antiderivative of $\boxed{f(x)\,.}$.

So, basically, if

$$\boxed{F'(x) \; = \; f(x)}\,,$$

then $\boxed{F(x)}$ is an antiderivative of $\boxed{f(x)}$.

- **Most basic antiderivatives.**

$$\begin{array}{cl}
x & \text{is an antiderivative of} \quad 1. \\
\frac{1}{2}x^2 & \text{is an antiderivative of} \quad x. \\
\frac{1}{3}x^3 & \text{is an antiderivative of} \quad x^2. \\
\frac{1}{4}x^4 & \text{is an antiderivative of} \quad x^3. \\
\frac{1}{5}x^5 & \text{is an antiderivative of} \quad x^4. \\
\frac{1}{6}x^6 & \text{is an antiderivative of} \quad x^5. \\
\frac{1}{7}x^7 & \text{is an antiderivative of} \quad x^6. \\
\frac{1}{8}x^8 & \text{is an antiderivative of} \quad x^7. \\
\frac{1}{9}x^9 & \text{is an antiderivative of} \quad x^8.
\end{array}$$

- **Antiderivative of a quantity that involves more than one term.**

Example. We know the fact

$$\text{The derivative of} \quad \frac{1}{3}x^3 + x \quad \text{is} \quad x^2 + 1.$$

Translation:

$$\frac{1}{3}x^3 + x \quad \text{is an antiderivative of} \quad x^2 + 1.$$

Example. We know the fact

$$\text{The derivative of} \quad \frac{2}{5}x^5 - 2x^3 + 4x^2 \quad \text{is} \quad 2x^4 - 6x^2 + 8x.$$

Translation:

$$\frac{2}{5}x^5 - 2x^3 + 4x^2 \quad \text{is an antiderivative of} \quad 2x^4 - 6x^2 + 8x.$$

- **Antiderivative of $f(x)$ is not unique. Two antiderivatives of $f(x)$ differ by a constant.**

Here is one important thing you should know.

" *If* $\boxed{F(x)}$ *is an antiderivative of* $\boxed{f(x)}$, *then*

$$F(x) + 1, \quad F(x) - 3, \quad F(x) + \frac{1}{2}, \quad F(x) - \sqrt{2}, \quad \textit{etc.}$$

too are all antiderivatives of the same $\boxed{f(x)}$."

Example.

$$\frac{1}{3}x^3 \quad \text{is an antiderivative of} \quad x^2.$$

$$\frac{1}{3}x^3 \; + \quad 1 \quad \text{is an antiderivative of} \quad x^2.$$

$$\frac{1}{3}x^3 \; + \quad 4 \quad \text{is an antiderivative of} \quad x^2.$$

$$\frac{1}{3}x^3 \; - \; 100 \; \text{is an antiderivative of} \quad x^2.$$

$$\frac{1}{3}x^3 \; + \; 256 \; \text{is an antiderivative of} \quad x^2.$$

$$\frac{1}{3}x^3 \; - \; \sqrt{2} \; \text{is an antiderivative of} \quad x^2.$$

$$\frac{1}{3}x^3 \; + \quad e \quad \text{is an antiderivative of} \quad x^2.$$

More generally, the following is true:

$$\frac{1}{3}x^3 \; + \; C \quad \text{is an antiderivative of} \quad x^2,$$

as long as C denotes a constant (real number).

- **Nature of antiderivatives.**

“$\underline{\underline{\textit{If}}}$ $\boxed{F(x)}$ $\underline{\underline{\textit{is one antiderivative of}}}$ $\boxed{f(x)}$, *then*

$\boxed{F(x) + C}$ $\left(\underline{\underline{C \text{ is an arbitrary constant real number}}} \right)$

$\underline{\underline{\textit{represents all the antiderivatives of}}}$ $\boxed{f(x)}$.”

Definition–Notation (Indefinite Integrals).

“$\underline{\underline{\text{Suppose}}}$ $\boxed{F(x)}$ $\underline{\underline{\text{is an antiderivative of}}}$ $\boxed{f(x)}$, $\underline{\underline{\text{then you write}}}$

$$\boxed{\int f(x)\,dx = F(x) + C}\ .$$

1. Always include “+ C” as a part of your answer.

2. Always include the tail-end “dx” inside the integral symbol.

Example.
$$\int x\,dx = \frac{1}{2}x^2 + C.$$

This means

“An antiderivative of x is $\frac{1}{2}x^2$.”

Example.
$$\int x^2\,dx = \frac{1}{3}x^3 + C.$$

This means

“An antiderivative of x^2 is $\frac{1}{3}x^3$.”

Q. Use the integral symbol to paraphrase

" An antiderivative of $4x^3$ is x^4."

$\left[\underline{\textbf{Answer}}\right]$**:**

$$\int 4\,x^3\,dx \quad = \quad x^4 \ + \ C.$$

Q. Use the integral symbol to paraphrase

"An antiderivative of $8x^3 + 8$ is $2x^4 + 8x$."

$\left[\underline{\textbf{Answer}}\right]$**:**

$$\int \left(8\,x^3 + 8\right) dx \quad = \quad 2\,x^4 \ + \ 8\,x \ + \ C.$$

- **Find indefinite integrals.**

Format. You are given a concrete $f\left(x\right)$, but you are not given its antiderivative $F\left(x\right)$. Find

$$\int f\left(x\right)\,dx.$$

Expect questions of this exact format.

$$\int 1\ dx \quad = \quad x \quad + \ C,$$

$$\int x\ dx \quad = \quad \frac{1}{2}x^2 \ + \ C,$$

$$\int x^2\ dx \quad = \quad \frac{1}{3}x^3 \ + \ C,$$

$$\int x^3\ dx \quad = \quad \frac{1}{4}x^4 \ + \ C,$$

$$\int x^4\ dx \quad = \quad \frac{1}{5}x^5 \ + \ C,$$

$$\vdots$$

Rule (monomial integration rule).

$$\boxed{\int x^n \, dx \quad = \quad \frac{1}{n+1} \; x^{n+1} \; + \; C}$$

$$\Big(n \quad \text{is an integer } \underline{\text{ constant }}; \qquad n \geq 0 \Big).$$

Example. Let's evaluate $\displaystyle\int \frac{1}{2} x^5 \, dx.$

You see $\frac{1}{2}$ inside the integral symbol. Pretend that there is no $\frac{1}{2}$. Then the integral would be $\frac{1}{6}x^6$, well, $+C$, but let's worry it later. So, $\frac{1}{6}x^6$. But that is when there is no $\frac{1}{2}$. In reality, there is $\frac{1}{2}$. So, the actual answer is

$$\frac{1}{2} \cdot \frac{1}{6} x^6 \;=\; \frac{1}{12} x^6.$$

Then don't forget $+C$. In sum:

$$\int \frac{1}{2} x^5 \, dx \;=\; \frac{1}{12} x^6 + C.$$

Example. Let's evaluate $\displaystyle\int \frac{7}{4} x^8 \, dx.$

You see $\frac{7}{4}$ inside the integral symbol. Pretend that there is no $\frac{7}{4}$. Then the integral would be $\frac{1}{9}x^9$. But that is when there is no $\frac{7}{4}$. In reality, there is $\frac{7}{4}$. So, the actual answer is

$$\frac{7}{4} \times \frac{1}{9} x^9 \;=\; \frac{7}{36} x^9.$$

Then don't forget $+C$. In sum:

$$\int \frac{7}{4} x^8 \, dx \;=\; \frac{7}{36} x^9 + C.$$

Example 8. Let's evaluate $\displaystyle\int \left(2x^4 + \frac{1}{4}x^2\right) dx.$

This is basically the same as the previous but notice that there are two terms in the integrand. The way it works is you do integrate term by term. First, $2x^4$ is integrated as $\dfrac{2}{5}x^5$. Next, $\dfrac{1}{4}x^2$ is integrated as $\dfrac{1}{12}x^3$. So, you construct the answer simply as

$$\frac{2}{5}x^5 + \frac{1}{12}x^3 + C.$$

In sum:

$$\int \left(2x^4 - \frac{1}{4}x^2\right) dx \;=\; \frac{2}{5}x^5 + \frac{1}{12}x^3 + C.$$

Example. Let's evaluate $\displaystyle\int \left(8x^7 - \frac{6}{5}x^5 + \frac{2}{7}x^3\right) dx.$

The same deal. Integrate term by term. Since

$$8x^7, \qquad \frac{6}{5}x^5 \qquad \text{and} \qquad \frac{2}{7}x^3$$

are integrated as

$$x^8, \qquad \frac{1}{5}x^6 \qquad \text{and} \qquad \frac{1}{14}x^4,$$

respectively, so

$$\int \left(8x^7 - \frac{6}{5}x^5 + \frac{2}{7}x^3\right) dx \;=\; x^8 - \frac{1}{5}x^6 + \frac{1}{14}x^4 + C.$$

Example. Let's evaluate $$\int \Big(x + 1\Big)\Big(x + 5\Big)\, dx.$$

You need to expand the integrand first. We know

$$\Big(x + 1\Big)\Big(x + 5\Big) = x^2 + 6x + 5.$$

Accordingly, the given integral becomes

$$\int \Big(x^2 + 6x + 5\Big)\, dx.$$

We know how to handle this. Namely,

$$\int \Big(x^2 + 6x + 5\Big)\, dx = \frac{1}{3}x^3 + 3x^2 + 5x + C.$$

Q. Evaluate

(1) $$\int 6\, x^3\, dx.$$

(2) $$\int \frac{1}{3}\, x^7\, dx.$$

(3) $$\int \Big(\frac{3}{4}\, x^2 + 3\Big)\, dx.$$

(4) $$\int \Big(\frac{8}{5}\, x^5 + 10\, x^4\Big)\, dx.$$

(5) $$\int \Big(x^2 + x + 1\Big)\, dx.$$

(6) $$\int \Big(x - 1\Big)\Big(x + 1\Big)\, dx.$$

(7) $$\int \Big(x + 1\Big)\Big(x + 3\Big)\, dx.$$

[**Answers**] :

(1) $\displaystyle \int 6\,x^3 \; dx \quad = \quad \frac{3}{2}\,x^4 \; + \; C.$

(2) $\displaystyle \int \frac{1}{3}\,x^7 \; dx \quad = \quad \frac{1}{24}\,x^8 \; + \; C.$

(3) $\displaystyle \int \Big(\frac{3}{4}\,x^2 \; + \; 3 \Big) \, dx \quad = \quad \frac{1}{4}\,x^3 \; + \; 3\,x \; + \; C.$

(4) $\displaystyle \int \Big(\frac{8}{5}\,x^5 \; + \; 10\,x^4 \Big) \, dx \quad = \quad \frac{4}{15}\,x^6 \; + \; 2\,x^5 \; + \; C.$

(5) $\displaystyle \int \Big(x^2 + x + 1 \Big) \, dx \quad = \quad \frac{1}{3}\,x^3 \; + \; \frac{1}{2}\,x^2 \; + \; x \; + \; C.$

(6) $\displaystyle \int \Big(x \, - \, 1 \Big) \Big(x \, + \, 1 \Big) \, dx \quad = \quad \frac{1}{3}\,x^3 \; - \; x \; + \; C.$

(7) $\displaystyle \int \Big(x \, + \, 1 \Big) \Big(x \, + \, 3 \Big) \, dx \quad = \quad \frac{1}{3}\,x^3 \; + \; 2\,x^2 \; + \; 3\,x \; + \; C.$

- §27. Bernoulli Polynomials and numbers – II.

Recall

Formula. Let n be a positive integer. Then

$$\boxed{\begin{aligned}
1 + 2 + 3 + 4 + \cdots + n &= \frac{1}{2}n^2 + \frac{1}{2}n\,,\\
1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\,,\\
1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2,\\
1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4 &= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n\,,\\
1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5 &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2,\\
1^6 + 2^6 + 3^6 + 4^6 + \cdots + n^6 &= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n\,.
\end{aligned}}$$

If you want to see more:

$$\boxed{1^k + 2^k + 3^k + 4^k + \cdots + n^k}$$

equals

$$\frac{1}{2}\,n^2 + \frac{1}{2}\,n, \qquad (k{=}1)$$

$$\frac{1}{3}\,n^3 + \frac{1}{2}\,n^2 + \frac{1}{6}\,n, \qquad (k{=}2)$$

$$\frac{1}{4}\,n^4 + \frac{1}{2}\,n^3 + \frac{1}{4}\,n^2, \qquad (k{=}3)$$

$$\frac{1}{5}\,n^5 + \frac{1}{2}\,n^4 + \frac{1}{3}\,n^3 - \frac{1}{30}\,n, \qquad (k{=}4)$$

$$\frac{1}{6}\,n^6 + \frac{1}{2}\,n^5 + \frac{5}{12}\,n^4 - \frac{1}{12}\,n^2, \qquad (k{=}5)$$

$$\frac{1}{7}\,n^7 + \frac{1}{2}\,n^6 + \frac{1}{2}\,n^5 - \frac{1}{6}\,n^3 + \frac{1}{42}\,n, \qquad (k{=}6)$$

$$\frac{1}{8}\,n^8 + \frac{1}{2}\,n^7 + \frac{7}{12}\,n^6 - \frac{7}{24}\,n^4 + \frac{1}{12}\,n^2, \qquad (k{=}7)$$

$$\frac{1}{9}\,n^9 + \frac{1}{2}\,n^8 + \frac{2}{3}\,n^7 - \frac{7}{15}\,n^5 + \frac{2}{9}\,n^3 - \frac{1}{30}\,n, \qquad (k{=}8)$$

$$\frac{1}{10}\,n^{10} + \frac{1}{2}\,n^9 + \frac{3}{4}\,n^8 - \frac{7}{10}\,n^6 + \frac{1}{2}\,n^4 - \frac{3}{20}\,n^2, \qquad (k{=}9)$$

$$\frac{1}{11}\,n^{11} + \frac{1}{2}\,n^{10} + \frac{5}{6}\,n^9 - n^7 + n^5 - \frac{1}{2}\,n^3 + \frac{5}{66}\,n, \qquad (k{=}10)$$

$$\frac{1}{12}\,n^{12} + \frac{1}{2}\,n^{11} + \frac{11}{12}\,n^{10} - \frac{11}{8}\,n^8 + \frac{11}{6}\,n^6 - \frac{11}{8}\,n^4 + \frac{5}{12}\,n^2, \qquad (k{=}11)$$

$$\frac{1}{13}\,n^{13} - \frac{1}{2}\,n^{12} + n^{11} - \frac{11}{6}\,n^9 + \frac{22}{7}\,n^7 - \frac{33}{10}\,n^5 + \frac{5}{3}\,n^3 - \frac{691}{2730}\,n\ . \qquad (k{=}12)$$

⋆ Switch from n to x:

- $\boldsymbol{f_k(x)}$.

$$\begin{aligned}
f_1\left(x\right) &= \frac{1}{2}\,x^2+\frac{1}{2}\,x,\\
f_2\left(x\right) &= \frac{1}{3}\,x^3+\frac{1}{2}\,x^2+\frac{1}{6}\,x,\\
f_3\left(x\right) &= \frac{1}{4}\,x^4+\frac{1}{2}\,x^3+\frac{1}{4}\,x^2,\\
f_4\left(x\right) &= \frac{1}{5}\,x^5+\frac{1}{2}\,x^4+\frac{1}{3}\,x^3-\frac{1}{30}\,x,\\
f_5\left(x\right) &= \frac{1}{6}\,x^6+\frac{1}{2}\,x^5+\frac{5}{12}\,x^4-\frac{1}{12}\,x^2,\\
f_6\left(x\right) &= \frac{1}{7}\,x^7+\frac{1}{2}\,x^6+\frac{1}{2}\,x^5-\frac{1}{6}\,x^3+\frac{1}{42}\,x,\\
f_7\left(x\right) &= \frac{1}{8}\,x^8+\frac{1}{2}\,x^7+\frac{7}{12}\,x^6-\frac{7}{24}\,x^4+\frac{1}{12}\,x^2,\\
f_8\left(x\right) &= \frac{1}{9}\,x^9+\frac{1}{2}\,x^8+\frac{2}{3}\,x^7-\frac{7}{15}\,x^5+\frac{2}{9}\,x^3-\frac{1}{30}\,x,\\
f_9\left(x\right) &= \frac{1}{10}\,x^{10}+\frac{1}{2}\,x^9+\frac{3}{4}\,x^8-\frac{7}{10}\,x^6+\frac{1}{2}\,x^4-\frac{3}{20}\,x^2,\\
f_{10}\left(x\right) &= \frac{1}{11}\,x^{11}+\frac{1}{2}\,x^{10}+\frac{5}{6}\,x^9-x^7+x^5-\frac{1}{2}\,x^3+\frac{5}{66}\,x,\\
f_{11}\left(x\right) &= \frac{1}{12}\,x^{12}+\frac{1}{2}\,x^{11}+\frac{11}{12}\,x^{10}-\frac{11}{8}\,x^8+\frac{11}{6}\,x^6-\frac{11}{8}\,x^4+\frac{5}{12}\,x^2,\\
f_{12}\left(x\right) &= \frac{1}{13}\,x^{13}-\frac{1}{2}\,x^{12}+x^{11}-\frac{11}{6}\,x^9+\frac{22}{7}\,x^7-\frac{33}{10}\,x^5+\frac{5}{3}\,x^3-\frac{691}{2730}\,x\ .
\end{aligned}$$

⋆ Now, “not for nothing”, let’s differentiate these:

- **Derivatives of $f_k(x)$.**

$$
\begin{aligned}
{f_1}'(x) &= x+\frac{1}{2}, \\
{f_2}'(x) &= x^2+x+\frac{1}{6}, \\
{f_3}'(x) &= x^3+\frac{3}{2}\,x^2+\frac{1}{2}\,x, \\
{f_4}'(x) &= x^4+2\,x^3+x^2-\frac{1}{30}, \\
{f_5}'(x) &= x^5+\frac{5}{2}\,x^4+\frac{5}{3}\,x^3-\frac{1}{6}\,x, \\
{f_6}'(x) &= x^6+3\,x^5+\frac{5}{2}\,x^4-\frac{1}{2}\,x^2+\frac{1}{42}, \\
{f_7}'(x) &= x^7+\frac{7}{2}\,x^6+\frac{7}{2}\,x^5-\frac{7}{6}\,x^3+\frac{1}{6}\,x, \\
{f_8}'(x) &= x^8+4\,x^7+\frac{14}{3}\,x^6-\frac{7}{3}\,x^4+\frac{2}{3}\,x^2-\frac{1}{30}, \\
{f_9}'(x) &= x^9+\frac{9}{2}\,x^8+6\,x^7-\frac{21}{5}\,x^5+2\,x^3-\frac{3}{10}\,x, \\
{f_{10}}'(x) &= x^{10}+5\,x^9+\frac{15}{2}\,x^8-7\,x^6+5\,x^4-\frac{3}{2}\,x^2+\frac{5}{66}, \\
{f_{11}}'(x) &= x^{11}+\frac{11}{2}\,x^{10}+\frac{55}{6}\,x^9-11\,x^7+11\,x^5-\frac{11}{2}\,x^3+\frac{5}{6}\,x, \\
{f_{12}}'(x) &= x^{12}+6\,x^{11}+11\,x^{10}-\frac{33}{2}\,x^8+22\,x^6-\frac{33}{2}\,x^4+5\,x^2-\frac{691}{2730}.
\end{aligned}
$$

Next, define (a) Bernoulli polynomials $B_k(x)$ (next page) and (b) $\underline{\text{truncated}}$ Bernoulli polynomials ${B_k}^\circ(x)$ (two pages later):

- **(a) Bernoulli polynomials $\boldsymbol{B_k(x)}$.**

$$B_1\left(x\right) = f_1{}'\Big(x-1\Big) = x - \frac{1}{2},$$

$$B_2\left(x\right) = f_2{}'\Big(x-1\Big) = x^2 - x + \frac{1}{6},$$

$$B_3\left(x\right) = f_3{}'\Big(x-1\Big) = x^3 - \frac{3}{2}\,x^2 + \frac{1}{2}\,x,$$

$$B_4\left(x\right) = f_4{}'\Big(x-1\Big) = x^4 - 2\,x^3 + x^2 - \frac{1}{30},$$

$$B_5\left(x\right) = f_5{}'\Big(x-1\Big) = x^5 - \frac{5}{2}\,x^4 + \frac{5}{3}\,x^3 - \frac{1}{6}\,x,$$

$$B_6\left(x\right) = f_6{}'\Big(x-1\Big) = x^6 - 3\,x^5 + \frac{5}{2}\,x^4 - \frac{1}{2}\,x^2 + \frac{1}{42},$$

$$B_7\left(x\right) = f_7{}'\Big(x-1\Big) = x^7 - \frac{7}{2}\,x^6 + \frac{7}{2}\,x^5 - \frac{7}{6}\,x^3 + \frac{1}{6}\,x,$$

$$B_8\left(x\right) = f_8{}'\Big(x-1\Big) = x^8 - 4\,x^7 + \frac{14}{3}\,x^6 - \frac{7}{3}\,x^4 + \frac{2}{3}\,x^2 - \frac{1}{30},$$

$$B_9\left(x\right) = f_9{}'\Big(x-1\Big) = x^9 - \frac{9}{2}\,x^8 + 6\,x^7 - \frac{21}{5}\,x^5 + 2\,x^3 - \frac{3}{10}\,x,$$

$$B_{10}\left(x\right) = f_{10}{}'\Big(x-1\Big) = x^{10} - 5\,x^9 + \frac{15}{2}\,x^8 - 7\,x^6 + 5\,x^4 - \frac{3}{2}\,x^2 + \frac{5}{66},$$

$$B_{11}\left(x\right) = f_{11}{}'\Big(x-1\Big) = x^{11} - \frac{11}{2}\,x^{10} + \frac{55}{6}\,x^9 - 11\,x^7 + 11\,x^5 - \frac{11}{2}\,x^3 - \frac{5}{6}\,x,$$

$$B_{12}\left(x\right) = f_{12}{}'\Big(x-1\Big) = x^{12} - 6\,x^{11} + 11\,x^{10} - \frac{33}{2}\,x^8 + 22\,x^6 - \frac{33}{2}\,x^4 + 5\,x^2 - \frac{691}{2730}.$$

- **(b) Truncated Bernoulli polynomials $\boldsymbol{B_k{}^\circ\big(x\big)}$.**

$$\begin{aligned}
B_1{}^\circ\,(x) &= 1 f_0\Big(x-1\Big) = x - 1,\\
B_2{}^\circ\,(x) &= 2 f_1\Big(x-1\Big) = x^2 - x,\\
B_3{}^\circ\,(x) &= 3 f_2\Big(x-1\Big) = x^3 - \frac{3}{2}\,x^2 + \frac{1}{2}\,x,\\
B_4{}^\circ\,(x) &= 4 f_3\Big(x-1\Big) = x^4 - 2\,x^3 + x^2,\\
B_5{}^\circ\,(x) &= 5 f_4\Big(x-1\Big) = x^5 - \frac{5}{2}\,x^4 + \frac{5}{3}\,x^3 - \frac{1}{6}\,x,\\
B_6{}^\circ\,(x) &= 6 f_5\Big(x-1\Big) = x^6 - 3\,x^5 + \frac{5}{2}\,x^4 - \frac{1}{2}\,x^2,\\
B_7{}^\circ\,(x) &= 7 f_6\Big(x-1\Big) = x^7 - \frac{7}{2}\,x^6 + \frac{7}{2}\,x^5 - \frac{7}{6}\,x^3 + \frac{1}{6}\,x,\\
B_8{}^\circ\,(x) &= 8 f_7\Big(x-1\Big) = x^8 - 4\,x^7 + \frac{14}{3}\,x^6 - \frac{7}{3}\,x^4 + \frac{2}{3}\,x^2,\\
B_9{}^\circ\,(x) &= 9 f_8\Big(x-1\Big) = x^9 - \frac{9}{2}\,x^8 + 6\,x^7 - \frac{21}{5}\,x^5 + 2\,x^3 - \frac{3}{10}\,x,\\
B_{10}{}^\circ\,(x) &= 10 f_9\Big(x-1\Big) = x^{10} - 5\,x^9 + \frac{15}{2}\,x^8 - 7\,x^6 + 5\,x^4 - \frac{3}{2}\,x^2,\\
B_{11}{}^\circ\,(x) &= 11 f_{10}\Big(x-1\Big) = x^{11} - \frac{11}{2}\,x^{10} + \frac{55}{6}\,x^9 - 11\,x^7 + 11\,x^5 - \frac{11}{2}\,x^3 - \frac{5}{6}\,x,\\
B_{12}{}^\circ\,(x) &= 12 f_{11}\Big(x-1\Big) = x^{12} - 6\,x^{11} + 11\,x^{10} - \frac{33}{2}\,x^8 + 22\,x^6 - \frac{33}{2}\,x^4 + 5\,x^2.
\end{aligned}$$

The circle '$\circ$' in the notation $B_k{}^\circ\,\big(x\big)$ indicates that it is different from $B_k\,\big(x\big)$.

Don't get confused between these two. The difference between $B_k\left(x\right)$ and $B_k{}^\circ\left(x\right)$ is only the constant term, and that constant is nothing but the k-th Bernoulli number B_k. So

$$\boxed{B_k\left(x\right) - B_k{}^\circ\left(x\right) = B_k.}$$

Or the same to say

$$\boxed{B_k\left(x\right) = B_k{}^\circ\left(x\right) + B_k.}$$

- **Initial conditions.**

The truncated Bernoulli polynomials satisfy some basic properties called initial conditions:

Initial conditions:

(i) $B_k{}^\circ\left(0\right) = 0$ for all k with $k \geq 2,$

(ii) $B_k{}^\circ\left(1\right) = 0$ for all k with $k \geq 1.$

The condition (i) simply means that $B_2{}^\circ\left(x\right),\ B_3{}^\circ\left(x\right),\ B_4{}^\circ\left(x\right),\ B_5{}^\circ\left(x\right),\ \cdots$ has no constant terms. These (i) and (ii) are important because they actually enable us to reconstruct the Bernoulli numbers and Bernoulli polynomials. To explain it, we need a new concept, called $\underline{\text{ antiderivatives }}$, which we will cover in the next lecture.

- Finally, the Bernoulli polynomials possess the following symmetry:

Symmetry:

$$B_k\left(1-x\right) = -B_k\left(x\right) \qquad \left(\text{when } k \text{ is odd}\right),$$

$$B_k\left(1-x\right) = B_k\left(x\right) \qquad \left(\text{when } k \text{ is even}\right).$$

- **Table of the first forty six (46) Bernoulli numbers.**

$$\begin{array}{lll lll}
B_1 & = & \frac{1}{2}, & B_2 & = & \frac{1}{6}, \\
B_3 & = & 0, & B_4 & = & \frac{-1}{30}, \\
B_5 & = & 0, & B_6 & = & \frac{1}{42}, \\
B_7 & = & 0, & B_8 & = & \frac{-1}{30}, \\
B_9 & = & 0, & B_{10} & = & \frac{5}{66}, \\
B_{11} & = & 0, & B_{12} & = & \frac{-691}{2730}, \\
B_{13} & = & 0, & B_{14} & = & \frac{7}{6}, \\
B_{15} & = & 0, & B_{16} & = & \frac{-3617}{510}, \\
B_{17} & = & 0, & B_{18} & = & \frac{43867}{798}, \\
B_{19} & = & 0, & B_{20} & = & \frac{-174611}{330}, \\
B_{21} & = & 0, & B_{22} & = & \frac{854513}{138}, \\
B_{23} & = & 0, & B_{24} & = & \frac{-236364091}{2730}, \\
B_{25} & = & 0, & B_{26} & = & \frac{8553103}{6}, \\
B_{27} & = & 0, & B_{28} & = & \frac{-23749461029}{870}, \\
B_{29} & = & 0, & B_{30} & = & \frac{8615841276005}{14322}, \\
B_{31} & = & 0, & B_{32} & = & \frac{-7709321041217}{510}, \\
B_{33} & = & 0, & B_{34} & = & \frac{2577687858367}{6}, \\
B_{35} & = & 0, & B_{36} & = & \frac{-26315271553053477373}{1919190}, \\
B_{37} & = & 0, & B_{38} & = & \frac{2929993913841559}{6}, \\
B_{39} & = & 0, & B_{40} & = & \frac{-261082718496449122051}{13530}, \\
B_{41} & = & 0, & B_{42} & = & \frac{1520097643918070802691}{1806}, \\
B_{43} & = & 0, & B_{44} & = & \frac{-27833269579301024235023}{690}, \\
B_{45} & = & 0, & B_{46} & = & \frac{596451111593912163277961}{282}.
\end{array}$$

- **How do we reconstruct Bernoulli polynomials from Bernoulli numbers?**

If you have followed "Review of Lectures – XXVI", 'Strategy C', and also what we have covered today, then you realize that the k-th Bernoulli polynomial $B_k(x)$ is reconstructed from

$$B_1,\ B_2,\ B_3,\ B_4,\ \cdots\ ,\ B_k,$$

as follows:

Process. First, binomially expand

$$\Big(x - B\Big)^k.$$

Then 'lower' the exponents for B, namely, replace B^1 with B_1; B^2 with B_2; B^3 with B_3, and so on. The outcome is $B_k\left(x\right)$.

If you want to get $B_k{}^\circ\left(x\right)$, then just subtract B_k from $B_k\left(x\right)$.

⋆ The reason you see the negative sign inside the parenthesis $\Big(x - B\Big)^k$ instead of the positive sign is essentially due to the shift $x \mapsto x-1$, made in the definition of $B_k\left(x\right)$ in page 6.

Example 1. Let's reconstrct $B_6\left(x\right)$ and $B_6{}^\circ\left(x\right)$ using this method, and using the table in page 9.

Step 1. Binomially expand

$$\Big(x - B\Big)^6.$$

The result is

$$x^6 - \binom{6}{1}B^1x^5 + \binom{6}{1}B^2x^4 - \binom{6}{1}B^3x^3 + \binom{6}{1}B^4x^2 - \binom{6}{1}B^5x + B^6.$$

Step 2. Lower the exponents for B:

$$x^6 - \binom{6}{1}B_1x^5 + \binom{6}{2}B_2x^4 - \binom{6}{3}B_3x^3 + \binom{6}{4}B_4x^2 - \binom{6}{5}B_5x + B_6.$$

Step 3. Throw concrete numbers for B_1, B_2, B_3, B_1, B_5 and B_6, using the table in page 9, and also throw concrete numbers for the binomial coefficients:

$$\begin{aligned} & x^6 - 6\cdot\frac{1}{2}x^5 + 15\cdot\frac{1}{6}x^4 - 20\cdot 0x^3 + 15\cdot\frac{-1}{30}x^2 - 6\cdot 0x + \cdot\frac{1}{42} \\ & \quad = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}. \end{aligned}$$

This is $B_6\left(x\right)$. In sum:

$$B_6\left(x\right) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}.$$

Step 4. As for $B_6{}^\circ\left(x\right)$, just frop the constant term, so

$$B_6{}^\circ\left(x\right) = x^6 - 3x^5 + x^4 - \frac{1}{2}x^2.$$

$\star$ You can write your answer in the following way:

$$\begin{aligned} B_6\left(x\right) &= x^6 - \binom{6}{1}B_1x^5 + \binom{6}{2}B_2x^4 - \binom{6}{3}B_3x^3 + \binom{6}{4}B_4x^2 - \binom{6}{5}B_5x + B_6 \\ &= x^6 - 6\cdot\frac{1}{2}x^5 + 15\cdot\frac{1}{6}x^4 - 20\cdot 0x^3 + 15\cdot\frac{-1}{30}x^2 - 6\cdot 0x + \frac{1}{42} \\ &= x^6 - 3x^5 + x^4 - \frac{1}{2}x^2 + \frac{1}{42}, \end{aligned}$$

$$B_6{}^\circ\left(x\right) = x^6 - 3x^5 + x^4 - \frac{1}{2}x^2.$$

Q. Mimic Example 1 above and reconstruct each of

(1) $B_9\left(x\right)$, $B_9{}^{\circ}\left(x\right)$. (2) $B_{12}\left(x\right)$, $B_{12}{}^{\circ}\left(x\right)$.

[**Solutions**]:

(1)
$$\begin{aligned}
B_9\left(x\right) &= x^9 - \binom{9}{1}B_1x^8 + \binom{9}{2}B_2x^7 - \binom{9}{3}B_3x^6 + \binom{9}{4}B_4x^5 - \binom{9}{5}B_5x^4 \\
&\qquad + \binom{9}{6}B_6x^3 - \binom{9}{7}B_7x^2 + \binom{9}{8}B_8x - B_9 \\
&= x^9 - 9\cdot\frac{1}{2}x^8 + 36\cdot\frac{1}{6}x^7 - 84\cdot 0x^6 + 126\cdot\frac{-1}{30}x^5 \\
&\qquad - 126\cdot 0x^4 + 84\cdot\frac{1}{42}x^3 - 36\cdot 0x^2 + 9\cdot\frac{-1}{30}x - 0 \\
&= x^9 - \frac{9}{2}x^8 + 6x^7 - \frac{21}{5}x^5 + 2x^3 - \frac{3}{10}x.
\end{aligned}$$

$$B_9{}^{\circ}\left(x\right) = x^9 - \frac{9}{2}x^8 + 6x^7 - \frac{21}{5}x^5 + 2x^3 - \frac{3}{10}x.$$

(2)
$$\begin{aligned}
B_{12}\left(x\right) &= x^{12} - \binom{12}{1}B_1x^{12} + \binom{12}{2}B_2x^{11} - \binom{12}{3}B_3x^{10} + \binom{12}{4}B_4x^9 \\
&\qquad - \binom{12}{5}B_5x^8 + \binom{12}{6}Bx^7 - \binom{12}{7}B_7x^6 + \binom{12}{8}B_8x^5 \\
&\qquad - \binom{12}{7}B_9x^4 + \binom{12}{8}B_{10}x^3 - \binom{12}{8}B_{11}x^2 + B_{12} \\
&= x^{12} - 12\cdot\frac{1}{2}x^{11} + 66\cdot\frac{1}{6}x^{10} - 220\cdot 0x^9 + 495\cdot\frac{-1}{30}x^8 \\
&\qquad - 792\cdot 0x^7 + 924\cdot\frac{1}{42}x^6 - 792\cdot 0x^5 + 495\cdot\frac{-1}{30}x^4 - 220\cdot 0x^3 \\
&\qquad + 66\cdot\frac{5}{66}x^2 - 12\cdot 0x + \frac{-691}{2730}
\end{aligned}$$

$$= \quad x^{12} - 6\,x^{11} + 11\,x^{10} - \frac{33}{2}\,x^{8} + 22\,x^{6} - \frac{33}{2}\,x^{4} + 5\,x^{2} - \frac{691}{2730}.$$

$$B_{12}{}^{\circ}\left(x\right) \;=\; x^{12} - 6\,x^{11} + 11\,x^{10} - \frac{33}{2}\,x^{8} + 22\,x^{6} - \frac{33}{2}\,x^{4} + 5\,x^{2}.$$

• §29. Applications of antiderivatives.

We have an alternative method to recover Bernoulli polynomials and numbers. Once again:

$$\begin{aligned}
&B_1{}^\circ(x) = x - 1, && B_1(x) = x - \frac{1}{2},\\
&B_2{}^\circ(x) = x^2 - x, && B_2(x) = x^2 - x + \frac{1}{6},\\
&B_3{}^\circ(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, && B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,\\
&B_4{}^\circ(x) = x^4 - 2x^3 + x^2, && B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},\\
&B_5{}^\circ(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, && B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x,\\
&B_6{}^\circ(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2, && B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42},\\
&B_7{}^\circ(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x, && B_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x,\\
&\qquad\qquad\vdots
\end{aligned}$$

Here is the recipé how to recover these one by one, from the complete scratch.

1. Starting point of construction. $B_1{}^\circ\left(x\right)$.

We do not rely on any prior knowledge about the shape of Bernoulli polynomials/ numbers, except

Initial conditions:

(i) $$B_k{}^\circ\left(0\right) \;=\; 0 \qquad \text{for all } k \text{ with} \qquad k \geq 2,$$

(ii) $$B_k{}^\circ\left(1\right) \;=\; 0 \qquad \text{for all } k \text{ with} \qquad k \geq 1,$$

and

(1d) $$B_1{}^\circ\left(x\right) \;=\; x \,-\, 1.$$

(Note that there is no (1a), (1b) and (1c).)

We are going to follow the step-by-step procedure below. Those steps exhibit the same pattern, each step uses the outcome of the previous step. So after three or so steps, you get the firm idea how to proceed. As you can see, in (2d), (3d), and (4d), you see that $B_2{}^\circ\left(x\right)$, $B_3{}^\circ\left(x\right)$, $B_4{}^\circ\left(x\right)$ are recovered. At the same time, In (2c), (3c), and (4c), you see that n B_1, B_2, and B_3 are recovered.

2. Determining B_1 and $B_2{}^\circ\left(x\right)$.

Add B_1 to (1d), and thereby make it $B_1\left(x\right)$. At this point B_1 is an unknown constant.

$$B_1\left(x\right) \;=\; x \,-\, 1 \,+\, B_1.$$

Take its antiderivative:

$$\int B_1\left(x\right) dx \;=\; \frac{1}{2}x^2 \,-\, x \,+\, B_1 \cdot x \,+\, C.$$

This is $\dfrac{1}{2} B_2{}^\circ\left(x\right)$, so

$$\frac{1}{2}B_2{}^{\circ}\left(x\right) \;=\; \frac{1}{2}x^2 \,-\, x \,+\, B_1 \cdot x \,+\, C. \tag{2a}$$

Substitute $x = 0,$ and $x = 1$ into (2a) independently. Use $B_2{}^{\circ}\left(0\right) = 0$ and $B_2{}^{\circ}\left(1\right) = 0$ $\big($'initial conditions'$\big)$:

$$\left\{\begin{aligned} 0 &= \frac{1}{2}\cdot 0^2 - 0 + B_1 \cdot 0 + C, \\ 0 &= \frac{1}{2}\cdot 1^2 - 1 + B_1 \cdot 1 + C. \end{aligned}\right. \tag{2b}$$

The first of the two equations in (2b) reads $0 = C$. So $C = 0$. Taking this into account, the second of the two equations in (2b) becomes

$$0 \;=\; \frac{1}{2} \,-\, 1 \,+\, B_1.$$

Solve it:

$$\begin{aligned} B_1 &= -\left(\frac{1}{2} - 1\right) \\ &= \frac{1}{2}. \end{aligned}$$

So we just found B_1. Namely,

$$B_1 \;=\; \frac{1}{2}. \tag{2c}$$

Substitute this and $C = 0$ back into (2a) above:

$$\begin{aligned} \frac{1}{2}B_2{}^{\circ}\left(x\right) &= \frac{1}{2}x^2 - x + \frac{1}{2}x \\ &= \frac{1}{2}x^2 - \frac{1}{2}x. \end{aligned}$$

Multiply 2 to the both sides:

$$B_2{}^{\circ}\left(x\right) \;=\; x^2 \,-\, x. \tag{2d}$$

3. Determining B_2 and $B_3^\circ(x)$.

Add B_2 to (2d), and thereby make it $B_2(x)$. At this point B_2 is an unknown constant.

$$B_2(x) = x^2 - x + B_2.$$

Take its antiderivative:

$$\int B_2(x)\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + B_2 \cdot x + C.$$

This is $\frac{1}{3}B_3^\circ(x)$, so

$$\text{(3a)} \qquad \frac{1}{3}B_3^\circ(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + B_2 \cdot x + C.$$

Substitute $x = 0$, and $x = 1$ into (3a) independently. Use $B_3^\circ(0) = 0$ and $B_3^\circ(1) = 0$ ('initial conditions'):

$$\text{(3b)} \qquad \begin{cases} 0 = \dfrac{1}{3} \cdot 0^3 - \dfrac{1}{2} \cdot 0^2 + B_2 \cdot 0 + C, \\[2ex] 0 = \dfrac{1}{3} \cdot 1^3 - \dfrac{1}{2} \cdot 1^2 + B_2 \cdot 1 + C. \end{cases}$$

The first of the two equations in (3b) reads $0 = C$. So $C = 0$. Taking this into account, the second of the two equations in (3b) becomes

$$0 = \frac{1}{3} - \frac{1}{2} + B_2.$$

Solve it:

$$\begin{aligned} B_2 &= -\left(\frac{1}{3} - \frac{1}{2}\right) \\ &= \frac{1}{6}. \end{aligned}$$

So we just found B_2. Namely,

(3c)
$$B_2 \ = \ \frac{1}{6}.$$

Substitute this and $C = 0$ back into (3a) above:

$$\frac{1}{3} B_3{}^{\circ}\left(x\right) \ = \ \frac{1}{3} x^3 \ - \ \frac{1}{2} x^2 \ + \ \frac{1}{6} x.$$

Multiply 3 to the both sides:

(3d)
$$B_3{}^{\circ}\left(x\right) \ = \ x^3 \ - \ \frac{3}{2} x^2 \ + \ \frac{1}{2} x.$$

4. Determining B_3 and $B_4{}^{\circ}\left(x\right)$.

Add B_3 to (3d), and thereby make it $B_3\left(x\right)$. At this point B_3 is an unknown constant.

$$B_3\left(x\right) \ = \ x^3 \ - \ \frac{3}{2} x^2 \ + \ \frac{1}{2} x \ + \ B_3.$$

Take its antiderivative:

$$\int B_3\left(x\right) dx \ = \ \frac{1}{4} x^4 \ - \ \frac{1}{2} x^3 \ + \ \frac{1}{4} x^2 \ + \ B_3 \cdot x \ + \ C.$$

This is $\frac{1}{4} B_4{}^{\circ}\left(x\right)$, so

(4a)
$$\frac{1}{4} B_4{}^{\circ}\left(x\right) \ = \ \frac{1}{4} x^4 \ - \ \frac{1}{2} x^3 \ + \ \frac{1}{4} x^2 \ + \ B_3 \cdot x \ + \ C.$$

Substitute $x = 0$, and $x = 1$ into (4a) independently. Use $B_4{}^{\circ}\left(0\right) = 0$ and $B_4{}^{\circ}\left(1\right) = 0$ (‘initial conditions’):

$$(4b) \qquad \begin{cases} 0 \; = \; \dfrac{1}{4} \cdot 0^4 - \dfrac{1}{2} \cdot 0^3 + \dfrac{1}{4} \cdot 0^2 + B_3 \cdot 0 + C, \\ 0 \; = \; \dfrac{1}{4} \cdot 1^4 - \dfrac{1}{2} \cdot 1^3 + \dfrac{1}{4} \cdot 1^2 + B_3 \cdot 1 + C. \end{cases}$$

The first of the two equations in (4b) reads $0 = C$. So $C = 0$. Taking this into account, the second of the two equations in (4b) becomes

$$0 \; = \; \frac{1}{4} - \frac{1}{2} + \frac{1}{4} + B_3.$$

Solve it:

$$\begin{aligned} B_3 \; &= \; -\left(\frac{1}{4} - \frac{1}{2} + \frac{1}{4}\right) \\ &= \; 0. \end{aligned}$$

So we just found B_3. Namely,

$$(4c) \qquad B_3 \; = \; 0.$$

Substitute this and $C = 0$ back into (4a) above:

$$\frac{1}{4} B_4{}^{\circ}\left(x\right) \; = \; \frac{1}{4} x^4 - \frac{1}{2} x^3 + \frac{1}{4} x^2.$$

Multiply 4 to the both sides:

$$(4d) \qquad B_4{}^{\circ}\left(x\right) \; = \; x^3 - 2x^3 + x^2.$$

$\star$ I think you have seen enough so you can take it from here. Namely, add B_4 to (4d) and then integrate it. That is $\dfrac{1}{5}$ of $B_5{}^{\circ}\left(x\right)$. One of the initial conditions: $B_5{}^{\circ}\left(0\right) = 0$ tells you $C = 0$. The other initial condition: $B_5{}^{\circ}\left(1\right) = 0$ allows you to determine B_4. Substituting it in the expression of $B_5{}^{\circ}\left(x\right)$ previously obtained which involves B_4 yields the actual shape of $B_5{}^{\circ}\left(x\right)$. So, in principle, as you keep going this way, you recover as many Bernoulli polynomials/numbers as you want, though in practice the computation gets longer and longer as you go on. Now you can do the exercise below.

Q. Assume

(7d)
$$B_7{}^\circ(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x,$$

and recover B_7 and $B_8{}^\circ(x)$.

$\Big[\underline{\textbf{Solution}}\Big]$**: Add B_7 to (7d), and thereby make it $B_7(x)$. At this point B_7 is an unknown constant.**

$$B_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x + B_7.$$

Take its antiderivative:

$$\int B_7(x)\,dx = \frac{1}{8}x^8 - \frac{1}{2}x^7 + \frac{7}{12}x^6 - \frac{7}{24}x^4 + \frac{1}{12}x^2 + B_7\cdot x + C.$$

This is $\frac{1}{8}B_8{}^\circ(x)$, **so**

(8a)
$$\frac{1}{8}B_8{}^\circ(x) = \frac{1}{8}x^8 - \frac{1}{2}x^7 + \frac{7}{12}x^6 - \frac{7}{24}x^4 + \frac{1}{12}x^2 + B_7\cdot x + C.$$

Substitute $x = 0$, and $x = 1$ into (8a) independently. Use $B_8{}^\circ(0) = 0$ and $B_8{}^\circ(1) = 0$ ('initial conditions'):

(8b)
$$\begin{cases} 0 = \frac{1}{8}0^8 - \frac{1}{2}0^7 + \frac{7}{12}0^6 - \frac{7}{24}0^4 + \frac{1}{12}0^2 + B_7\cdot 0 + C. \\ 0 = \frac{1}{8}1^8 - \frac{1}{2}1^7 + \frac{7}{12}1^6 - \frac{7}{24}1^4 + \frac{1}{12}1^2 + B_7\cdot 1 + C. \end{cases}$$

The first of the two equations in (8b) reads $0 = C$. So $C = 0$. Taking this into account, the second of the two equations in (8b) becomes

$$0 = \frac{1}{8} - \frac{1}{2} + \frac{7}{12} - \frac{7}{24} + \frac{1}{12} + B_7.$$

Solve it:

$$\begin{aligned} B_7 &= -\left(\frac{1}{8} - \frac{1}{2} + \frac{7}{12} - \frac{7}{24} + \frac{1}{12}\right) \\ &= 0. \end{aligned}$$

So we just found B_0. Namely,

(8c) $$B_7 = 0.$$

Substitute this and $C = 0$ back into (8a) above:

$$\frac{1}{8} B_8{}^{\circ}\left(x\right) = \frac{1}{8}x^8 - \frac{1}{2}x^7 + \frac{7}{12}x^6 - \frac{7}{24}x^4 + \frac{1}{12}x^2.$$

Multiply 8 to the both sides:

(8d) $$B_8{}^{\circ}\left(x\right) = x^8 - 4x^7 + \frac{14}{3}x^6 - \frac{7}{3}x^4 + \frac{2}{3}x^2.$$

Q. **Assume**

(12d) $$B_{12}{}^{\circ}\left(x\right) = x^{12} - 6\,x^{11} + 11\,x^{10} - \frac{33}{2}\,x^8 + 22\,x^6 - \frac{33}{2}\,x^4 + 5\,x^2,$$

and recover B_{12} and $B_{13}{}^{\circ}(x)$.

[Answer]:

$$B_{12} = \frac{-691}{2730},$$

$$B_{13}\left(x\right) = x^{13} - \frac{13}{2}x^{12} + 13x^{11} - \frac{143}{6}x^9 + \frac{286}{7}x^7 - \frac{429}{10}x^5 + \frac{65}{3}x^3 - \frac{691}{210}x.$$