Math 105 TOPICS IN MATHEMATICS

STUDY GUIDE FOR FINAL EXAM – FA

May 1 (Fri), 2015

Instructor: Yasuyuki Kachi

Line #: 52920.

- **§18. Exponential functions.**

- Exponential functions are functions such as

$$2^x, \qquad e^x, \qquad 3^x, \qquad 4^x, \qquad \cdots .$$

Definition. Assume that a is a positive real number, x is a real number, and ℓ is an integer, $\ell > 1$. For each index n, let $c_n\big(\ell\big)$ be the truncation at the n-th digit under the "ℓ-ary" point of x. Define

$$\boxed{a^x \;=\; \lim_{n \to \infty} a^{c_n(\ell)}} \; .$$

This limit exists, and it does not depend on the choice of ℓ.

Exponential Laws (refined). Let x and y be real numbers. Let a and b be positive real numbers Then

Rule I. $$\boxed{\big(ab\big)^x \;=\; a^x\, b^x} \; .$$

Rule II. $$\boxed{a^x\, a^y \;=\; a^{x+y}} \; .$$

Rule III. $$\boxed{\big(a^x\big)^y \;=\; a^{xy}} \; .$$

Rule IV. $$\boxed{a^0 \;=\; 1} \; , \qquad \boxed{1^x \;=\; 1} \; .$$

Rule V. $$\boxed{a^{-x} = \frac{1}{a^x}}\ .$$

Q.

(1) Simplify $2^x \cdot 5^x$. Write your answer as in $\boxed{}^{x}$.

(2) Simplify $a^3 \cdot a^8$.

(3) Simplify $\left(a^{\sqrt{2}}\right)^{\sqrt{2}}$.

(4) Simplify $1^{\sqrt{3}}$.

(5) Rewrite $a^{-\sqrt{5}}$ in the form $\dfrac{1}{\boxed{}}$.

[**Answers**]: (1) 10^x. (2) a^{11}. (3) a^2. (4) 1. (5) $\dfrac{1}{a^{\sqrt{5}}}$.

- $\boldsymbol{e^x}$**.**

Now, among all exponential functions a^x, the one with $a = e$ has a very very special place. Often when we say "the exponential function", it refers to e^x. Here is the reason why:

Theorem. Let x be an arbitrary real number. Then

$$\boxed{\begin{aligned} e^x &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\ &= \lim_{k\to\infty}\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\cdots+\frac{1}{k!}x^k\right). \end{aligned}}$$

- **Notational remark.** We often write

$$\lim_{k\to\infty}\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\cdots+\frac{1}{k!}x^k\right)$$

as

$$1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\cdots \quad .$$

If you incorporate this notation, then the above theorem is paraphrased as follows:

Theorem paraphrased. Let x be an arbitrary real number. Then

$$\boxed{\begin{aligned} e^x &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\ &= 1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\cdots \end{aligned}}$$

Example 1.

$$\sqrt{e} = 1 + \frac{1}{1!} \cdot \frac{1}{2} + \frac{1}{2!} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{3!} \cdot \left(\frac{1}{2}\right)^3 + \frac{1}{4!} \cdot \left(\frac{1}{2}\right)^4 + \frac{1}{5!} \cdot \left(\frac{1}{2}\right)^5 + \frac{1}{6!} \cdot \left(\frac{1}{2}\right)^6 + \cdots .$$

- **Exponential Laws pertaining to e^x.**

Rule II. $$\boxed{e^x\, e^y = e^{x+y}} .$$

Rule III. $$\boxed{\left(e^x\right)^y = e^{xy}} .$$

Rule IV. $$\boxed{e^0 = 1} .$$

Rule V. $$\boxed{e^{-x} = \frac{1}{e^x}} .$$

Q. Find the limits:

(1) $$\lim_{n \to \infty} \left(1 + \frac{3}{n}\right)^n = ?$$

(2) $$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = ?$$

(3) $$\lim_{n \to \infty} \left(1 - \frac{\sqrt{2}}{n}\right)^n = ?$$

$\left[\underline{\textbf{Answers}}\right]$**:** (1) e^3. (3) e^{-1}. (2) $e^{-\sqrt{2}}$.

Q. Write up each of (1) e^2, (1) $\sqrt[3]{e}$, and (3) e^{-1} as an infinite sum in the same fashion as Example 1.

$\left[\underline{\textbf{Answers}}\right]$**:**

(1) $$e^2 = 1 + \frac{1}{1!} \cdot 2 + \frac{1}{2!} \cdot 2^2 + \frac{1}{3!} \cdot 2^3 + \frac{1}{4!} \cdot 2^4 + \cdots .$$

(2) $$\sqrt[3]{e} = 1 + \frac{1}{1!} \cdot \frac{1}{3} + \frac{1}{2!} \cdot \left(\frac{1}{3}\right)^2 + \frac{1}{3!} \cdot \left(\frac{1}{3}\right)^3 + \frac{1}{4!} \cdot \left(\frac{1}{3}\right)^4 + \frac{1}{5!} \cdot \left(\frac{1}{3}\right)^5 + \cdots .$$

(3) $$e^{-1} = 1 + \frac{1}{1!} \cdot \left(-1\right) + \frac{1}{2!} \cdot \left(-1\right)^2 + \frac{1}{3!} \cdot \left(-1\right)^3 + \frac{1}{4!} \cdot \left(-1\right)^4 + \cdots$$

$$\left(= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots\right).$$

- **§19. Logarithm.**

⋆ Where do ‘log’s show up? Let’s go back to

$$2^x, \qquad e^x, \qquad 3^x, \qquad 4^x, \qquad \cdots \ .$$

Pop quiz. Can you fill in the boxes?

$$2^x = 6^{\boxed{}} .$$

$$10^x = e^{\boxed{}} .$$

[**Answers**]:

$$2^x = 6^{\boxed{(\log_6 2)\, x}} .$$

$$10^x = e^{\boxed{(\ln 10)\, x}} .$$

So, the first role of 'log' is it serves as a 'buffer', to go from one exponential function to another $\big($such as going from 2^x to $3^x\big)$. And, if you realize, this actually pretty much tells you what 'log's are.

Indeed, substitute $x = 1$ into

$$\boxed{a^x = b^{(\log_b a)\, x}}$$

and get

$$\boxed{a = b^{\log_b a}}\,.$$

So, the bottm line of what 'log' is is summarized in one line:

"$\boxed{x = \log_b a}$ $\underline{\underline{\textit{is a number satisfying}}}$ $\boxed{b^x = a}$".

Pop quiz. Can you fill in the boxes?

$$2 = 6^{\boxed{}}\,.$$

$$10 = e^{\boxed{}}\,.$$

[**Answers**]:

$$2 = 6^{\boxed{\log_6 2}}\,.$$

$$10 = e^{\boxed{\ln 10}}\,.$$

⋆ There is one thing you might wonder at this stage:

- ○ the relationship between $\log_2 3$ and $\log_3 2$,
- ○ the relationship between $\log_3 4$ and $\log_4 3$,
- ○ the relationship between $\log_7 5$ and $\log_5 7$,

and so on. This is simple:

$$\log_3 2 = \frac{1}{\log_2 3},$$

$$\log_3 4 = \frac{1}{\log_4 3},$$

$$\log_7 5 = \frac{1}{\log_5 7}.$$

More generally:

Fact. Let a and b be positive real numbers. Then

$$\boxed{\log_b a = \frac{1}{\log_a b}}.$$

Pop quiz. Write each of the following in the form

$$\log_{\square} \boxed{}.$$

(a) $\dfrac{1}{\log_3 11}$. (b) $\dfrac{1}{\log_{10} 24}$. (c) $\dfrac{1}{\ln 5}$.

[**Answers**]: (a) $\log_{11} 3$. (b) $\log_{24} 10$. (c) $\log_5 e$.

Q 1. $\log_2 2048 \;=?$ $\quad \log_2 8192 \;=?$ $\quad \log_2 32768 \;=?$ $\quad \log_2 65536 \;=?$

Consult the table below, if necessary.

n	11	12	13	14	15	16
2^n	2048	4096	8192	16384	32768	65536

[**Answers**]: $\log_2 2048 \;=\; 11.$ $\qquad \log_2 28192 \;=\; 13.$

$$\log_2 32668 \;=\; 15. \qquad \log_2 65536 \;=\; 16.$$

Q 2. $\log_2 \dfrac{1}{2048} \;=?$ $\quad \log_2 \dfrac{1}{4096} \;=?$ $\quad \log_2 \dfrac{1}{16384} \;=?$

$$\log_2 \frac{1}{65536} \;=?$$

(Consult the table above if necessary.)

[**Answers**]: $\log_2 \dfrac{1}{2048} \;=\; -11.$ $\qquad \log_2 \dfrac{1}{4096} \;=\; -12.$

$$\log_2 \frac{1}{16384} \;=\; -14. \qquad \log_2 \frac{1}{65536} \;=\; -16.$$

Q 3. $\log_3 19683 \;=?$ $\quad\log_3 177147 \;=?$ $\quad\log_3 1594323 \;=?$

Consult the table below, if necessary.

n	9	10	11	12	13	14
3^n	19683	59049	177147	531441	1594323	4782969

[**Answers**]: $\log_3 19683 = 9.$ $\quad\log_3 177147 = 11.$

$\log_3 1594323 = 13.$

Q. $\log_{10} 1000000000 \;=?$ $\quad\big(1$ followed by nine straight 0s$\big)$.

$\log_{10} \dfrac{1}{1000000000000} \;=?$ $\quad\big($The denominator: 1 followed by twelve straight 0s$\big)$.

$\log_{10} 10^{24} \;=?$ $\quad\log_{10} 10^{-60} \;=?$

[**Answers**]: $\log_{10} 1000000000 = 9.$

$$\log_{10} \frac{1}{1000000000000} = -12.$$

$\log_{10} 10^{24} = 24.$ $\quad\log_{10} 10^{-60} = -60.$

- Recap:

$$\boxed{\log_{10} 10^n = n}\;.$$

Also, in retrospect,

$$\boxed{\log_2 2^n = n}$$

and

$$\boxed{\log_3 3^n = n}$$

More generally:

$$\boxed{\log_a a^n = n}$$

(where n is an integer).

$\star$ What's more, in this there is no reason n has to be an integer in order for the statement to be true.

$$\boxed{\log_a a^x \ = \ x}$$

(where x is a real number).

- **$\log_b 1$.** No matter what b is (provided b is a positive real number and $b \neq 1$), $\log_b 1$ always equal to 0.

$$\boxed{\log_b 1 \ = \ 0.}\ .$$

- **$\log_1 a$ is undefined.**

$\log_1 a$ is undefined. This is just because 1^x always equals 1, no matter what x is. Another way to see it is $\log_1 a$ would be the reciprocal of $\log_a 1$, but $\log_a 1$ equals 0. The reciprocal of 0 is undefined. (You might quibble that, because of $1^1 = 1$, we should say $\log_1 1 = 1$. True. However, someone else might argue that, because of $1^0 = 1$, we should say $\log_1 1 = 0$. The bottom line is, $\log_1 a$ for $a \neq 1$ is undefined, so, considering $\log_1 a$ as a function on a is pointless.)

- **$\log_0 a$ is undefined.**

$\log_0 a$ is undefined. This is just because 0^x always equals 0, no matter what x is (provided x is positive).

- **$\log_b 0$ is undefined.**

$\log_b 0$ is undefined. Actually, depending on a context, provided b is a positive real number and $b \neq 1$, $\log_b 0$ makes sense as a limit

$$\lim_{x \longrightarrow 0} \log_b x.$$

Let's not worry about this for now, though just in case

$$\lim_{x \longrightarrow 0} \log_b x \quad = \quad \begin{cases} -\infty & (b > 1), \\ +\infty & (b < 1). \end{cases}$$

- So, from now on, when we talk about $\boxed{\log_a b}$, we always assume

$$\boxed{a > 0, \qquad a \neq 1, \qquad \text{and} \qquad b > 0.}$$

In the future, whenever we write $\log_a b$, these conditions on a and b will be automatically assumed.

- **Summary.**

This is a good place to review two important things we have learned so far. One:

$$\text{“} \boxed{x = \log_b a} \quad \textit{is a number satisfying} \quad \boxed{b^x = a} \text{”}.$$

(This is from page 4.) In particular,

$$b^{\log_b a} = a \qquad (a > 0).$$

Two:

$$\log_a a^x = x \qquad (x \text{ is a real number}).$$

(This is from page 15.)

These are usually put together, and called cancellation laws:

- **Cancellation laws.**

$$\boxed{\begin{aligned} b^{\log_b a} &= a \qquad (a > 0). \\ \log_a a^x &= x \qquad (x \text{ is a real number}). \end{aligned}}$$

It is worthwhile to isolate the cancellation laws for the natural log 'ln':

- **Cancellation laws for 'ln'.**

$$e^{\ln a} = a \qquad \big(a > 0\big).$$

$$\ln e^{x} = x \qquad \big(x \text{ is a real number}\big).$$

Q. Use cancellation laws to simplify:

(1) $2^{\log_2 5}$. (2) $3^{\log_3 10}$. (3) $5^{\log_5 \frac{7}{3}}$. (4) $e^{\ln \sqrt{2}}$.

(5) $9^{\log_3 5}$. $\Big($Hint: $9 = 3 \cdot 3$, so $9^{\log_3 5} = 3^{\log_3 5} \cdot 3^{\log_3 5}$.$\Big)$

[**Answers**]: (1) 5. (2) 10. (3) $\frac{7}{3}$. (4) $\sqrt{2}$.

(5) 25.

Q. Use cancellation laws to simplify:

(1) $\log_3 3^6$. (2) $\log_2 2^{\frac{7}{2}}$. (3) $\log_{10} \sqrt{10}$. (4) $\ln e^{\pi}$.

(5) $\log_{49} 7$. $\Big($Hint: $7 = 49^{\frac{1}{2}}$.$\Big)$

[**Answers**]: (1) 6. (2) $\frac{7}{2}$. (3) $\frac{1}{2}$. (4) π.

(5) $\frac{1}{2}$.

- **Change of base.**

There are some important laws about 'log'. Just like the two exponential functions are related, two logarithms are related:

$$\log_b c \;=\; \frac{\log_a c}{\log_a b}.$$

In particular,

$$\log_b c \;=\; \frac{\ln c}{\ln b}.$$

Q. Simplify:

(1) $\dfrac{\log_3 7}{\log_3 4}$. (2) $\dfrac{\log_{11} 26}{\log_{11} 15}$. (3) $\dfrac{\ln 100}{\ln 10}$. (4) $\dfrac{\log_2 7}{\log_2 e}$.

Write the answer in the form

$$\log_{\square} \boxed{} \qquad \text{or} \qquad \ln \boxed{}$$

(which ever is applicable). If there is still a room for simplification, simplify.

[**Answers**]: (1) $\log_4 7$. (2) $\log_{15} 26$. (3) $\log_{10} 100 \;=\; 2$. (4) $\ln 7$.

- **Logarithmic Laws.**

Below (i), (ii) and (iii) are the logarithmic laws for 'ln'.

(i) $$\ln\Big(x\,y\Big) \;=\; \Big(\ln x\Big) + \Big(\ln y\Big)$$

$$\Big(x > 0, \quad y > 0\Big),$$

(ii) $$\ln \frac{x}{y} \;=\; \Big(\ln x\Big) - \Big(\ln y\Big)$$

$$\Big(x > 0, \quad y > 0\Big),$$

(iii) $$\ln\Big(x^a\Big) \;=\; a\Big(\ln x\Big)$$

$$\Big(x > 0\Big).$$

$\star$ Though there is no compelling reason to do so, just for once I want to use the symbols $\heartsuit$, $\diamondsuit$, $\clubsuit$ for x, y and z. It will give you a different impression:

(i) $$\ln\Big(\heartsuit\,\diamondsuit\Big) \;=\; \Big(\ln \heartsuit\Big) + \Big(\ln \diamondsuit\Big)$$

$$\Big(\heartsuit > 0, \quad \diamondsuit > 0\Big),$$

(ii) $$\ln \frac{\heartsuit}{\diamondsuit} \;=\; \Big(\ln \heartsuit\Big) - \Big(\ln \diamondsuit\Big)$$

$$\Big(\heartsuit > 0, \quad \diamondsuit > 0\Big),$$

(iii) $$\ln\Big(\heartsuit^{\clubsuit}\Big) \;=\; \clubsuit\Big(\ln \heartsuit\Big)$$

$$\Big(\heartsuit > 0\Big).$$

- Let's isolate the case $\clubsuit = \dfrac{1}{2}$ in (iii):

$$\ln\left(\heartsuit^{\frac{1}{2}}\right) = \frac{1}{2}\left(\ln\heartsuit\right) \qquad \left(\heartsuit > 0\right).$$

or the same

$$\ln\sqrt{\heartsuit} = \frac{1}{2}\left(\ln\heartsuit\right) \qquad \left(\heartsuit > 0\right).$$

Example. $\left(\ln 2\right) + \left(\ln 3\right)$ equals $\ln 6$. Note that

$$\left(\ln 2\right) + \left(\ln 3\right) \neq \ln 5.$$

Example. $\left(\ln 3\right) - \left(\ln 2\right)$ equals $\ln\dfrac{3}{2}$. Note that

$$\left(\ln 3\right) - \left(\ln 2\right) \neq \ln 1.$$

Example. $5\ \ln 2$ equals $\ln 32$. Note that

$$5\ \ln 2 \neq \ln 10.$$

Example. $\dfrac{1}{2}\ \ln 6$ equals $\ln\sqrt{6}$. Note that

$$\frac{1}{2}\ \ln 6 \neq \ln 3.$$

Example. Let's simplify

$$e^{(\ln 3) + (\ln 7)}.$$

We use (i) of the Logarithmic Laws, and Cancellation Laws. This quantity is of the form $e^{\heartsuit}$, where $\heartsuit = \Big(\ln 3 \Big) + \Big(\ln 7 \Big)$. By (i) of the Logarithmic Laws, this $\heartsuit$ equals $\ln 21$. Hence the original quantity $e^{\heartsuit}$ equals $e^{\ln 21}$. By Cancellation Laws, this quantity equals 21.

Example. Let's simplify

$$e^{3 \ \ln \ 2}.$$

We use (iii) of the Logarithmic Laws, and Cancellation Laws. This quantity is of the form $e^{\heartsuit}$, where $\heartsuit = 3 \ln 2$. By (iii) of the Logarithmic Laws, this $\heartsuit$ equals $\ln 8$. Hence the original quantity $e^{\heartsuit}$ equals $e^{\ln 8}$. By Cancellation Laws, this quantity equals 8.

Example. Let's simplify

$$\ln \ 2^{\frac{1}{\ln \ 2}}.$$

We use (iii) of the Logarithmic Laws. First, the above quantity is of the form $\ln \ \heartsuit^{\clubsuit}$ where $\heartsuit = 2$, and $\clubsuit = \dfrac{1}{\ln 2}$. By (iii) of the Logarithmic Laws, this quantity equals $\clubsuit \ \ln \heartsuit$, that is,

$$\frac{1}{\ln 2} \ \cdot \ \ln 2.$$

This is simplified to 1. So the answer is 1.

Example. Let's simplify

$$2^{\frac{1}{\ln \ 2}}.$$

As for this, we have just worked out in the previous example that, ' ln ' of the quantity $2^{\frac{1}{\ln \ 2}}$ equals 1. Hence the quantity $2^{\frac{1}{\ln \ 2}}$ itself equals e. So, the answer is e.

Exercise 9.

(1) Simplify $\Big(\ln 3\Big) + \Big(\ln 12\Big)$. Write your answer in the form $\ln \boxed{}$.

(2) Simplify $\Big(\ln 15\Big) - \Big(\ln 5\Big)$. Write your answer in the form $\ln \boxed{}$.

(3) Rewrite $4 \ln 3$ in the form $\ln \boxed{}$.

(4) Rewrite $\ln 256$ in the form $\boxed{}\Big(\ln 2\Big)$.

(5) Rewrite $\ln \sqrt[3]{7}$ in the form $\dfrac{1}{\boxed{}}\Big(\ln 7\Big)$.

(6) Rewrite $\ln \sqrt[5]{81}$ in the form $\dfrac{\boxed{}}{\boxed{}}\Big(\ln 3\Big)$.

(7) Simplify $e^{(\ln 4)+(\ln 13)}$.

(8) Simplify $e^{2\,(\ln 5)}$.

(9) Simplify $\ln 5^{\frac{1}{\ln 5}}$.

(10) Simplify $5^{\frac{1}{\ln 5}}$.

[**Answers**]: (1) $\ln 36$. (2) $\ln 3$. (3) $\ln 81$. (4) $8 \ln 2$.

(5) $\dfrac{1}{3} \ln 7$. (6) $\dfrac{4}{5} \ln 3$. (7) 52. (8) 25. (9) 1.

(10) e.

Exercise 10. Verify:

(1) $2^{\frac{x}{\ln 2}} = e^x.$

(2) $x^x = e^{x \ln x}.$

(3) $x^{\frac{1}{x}} = e^{\frac{\ln x}{x}}.$

(4) $2^x = e^{x (\ln 2)}.$

(5) $x^{\ln x} = e^{\left((\ln x)^2 \right)}.$

(6) $2^{\ln x} = e^{(\ln 2)(\ln x)} = x^{\ln 2}.$

(7) $\left(\ln x \right)^{\frac{1}{x}} = e^{\frac{\ln(\ln x)}{x}}.$

[**Solutions**]: Take 'ln' of the two sides, and verify that the resulting qualtities from the two sides are equal.

Exercise 11.

(1) Are $x^{\left(y^z\right)}$ and $\left(x^y\right)^z$ the same?

(2) Are $\left(x^y\right)^{(z^w)}$, $x^{\left(\left(y^z\right)^w\right)}$ and $x^{\left(y^{\left(z^w\right)}\right)}$ all the same?

[**Answers**]: (1) They are different. Indeed,

$$2^{\left(3^2\right)} = 2^9, \qquad \left(2^3\right)^2 = 8^2 = 2^6.$$

(2) They are all different. Indeed,

$$\left(2^3\right)^{(2^3)} = 8^8 = 2^{24}, \qquad 2^{\left(\left(3^2\right)^3\right)} = 2^{\left(9^3\right)} = 2^{729},$$

$$2^{\left(3^{\left(2^3\right)}\right)} = 2^{\left(3^8\right)} = 2^{6561}.$$

- §22. **Polynomials and their arithmetic.**

Q. Permute the order of terms, if necessary, o make each of the given polynomials in the ascending order.

(1) $2x + x^4 - \frac{1}{2}x^2$. (2) $-\frac{4}{3}x^3 - 5x^2 - 4x^5$.

(3) $x^8 + 5x^6 - 10x^9 + 3$. (4) $x + \sqrt{5}\,x^5 + \sqrt{3}\,x^3 - \sqrt{2}\,x^2$.

(5) $x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

[**Answers**]:

(1) $2x - \frac{1}{2}x^2 + x^4$. (2) $-5x^2 - \frac{4}{3}x^3 - 4x^5$.

(3) $3 + 5x^6 + x^8 - 10x^9$. (4) $x - \sqrt{2}\,x^2 + \sqrt{3}\,x^3 + \sqrt{5}\,x^5$.

(5) $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9$.

Q. Permute the order of terms, if necessary, to make each of the given polynomials in the descending order.

(1) $6x^2 + x^3$. (2) $\frac{1}{2}x^3 + \frac{1}{3}x^4 - x$.

(3) $x^7 + 5x^5 - 10x^8 + 4x^6$. (4) $1 - x^2 + x^4 - x^6 + x^8 - x^{10}$.

[**Answers**]:

(1) $x^3 + 6x^2$. (2) $\frac{1}{3}x^4 + \frac{1}{2}x^3 - x$.

(3) $-10x^8 + x^7 + 4x^6 + 5x^5$. (4) $-x^{10} + x^8 - x^6 + x^4 - x^2 + 1$.

Q. Do

(1) $\left(x^7 + 3\,x^5 + 2\,x^3\right) + \left(-\,x^6 - x^4 - 2\,x^2\right).$

(2) $\left(x^4 + 9\,x^3 + 1\right) + \left(-\,x^4 - x^3 - 5\,x^2 + 2x + 3\right).$

(3) $\left(\frac{1}{2}x^3 + \frac{1}{3}x\right) + \left(\frac{1}{3}x^3 - \frac{1}{4}x\right).$

(4) $f(x) + g(x),$ where

$$f(x) = x^6 + 8x^5 + 12x^4 + 36x^3 + 9x^2,$$

$$g(x) = x^8 - 3x^6 - 8x^4 - 24x^3 + 45x - 120.$$

(5) $f(x) + g(x),$ where

$$f(x) = x^7 + x^5 + x^3 + x,$$

$$g(x) = x^8 + x^6 + x^4 + x^2 + 1.$$

[**Answers**]**:**

(1) $x^7 - x^6 + 3x^5 - x^4 + 2x^3 - 2\,x^2.$

(2) $8\,x^3 - 5x^2 + 2x + 4.$

(3) $\frac{5}{6}\,x^3 + \frac{1}{12}\,x.$

(4) $x^8 - 2x^6 + 8x^5 + 4x^4 + 12x^3 + 9x^2 + 45x - 120.$

(5) $x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$

Q. Do

(1) $\Big(x^3 + 11\,x^2 + 21\,x\Big) - \Big(-\,x^2 - x + 4\Big).$

(2) $\Big(-\,x^7 + 5x^6 + x^3 - 6\Big) - \Big(-\,2x^6 - 3x^4 + 7\,x^3 + 2x + 5\Big).$

(3) $\left(\dfrac{3}{2}x^4 + \dfrac{7}{4}x^2\right) - \left(\dfrac{1}{6}x^4 - \dfrac{1}{4}x^2 + 1\right).$

(4) $f\,(x) - g\,(x),$ where

$$f\,(x) = x^5 + 4x^4 + 16x^3 + 22x^2 + 18x,$$
$$g\,(x) = x^6 - 31x^4 - 62x^2 + 72x + 56.$$

(5) $f\,(x) - g\,(x),$ where

$$f\,(x) = x^{13} + x^9 + x^5 + x, \qquad g\,(x) = x^{11} + x^7 + x^3.$$

[**Answers**]:

(1) $x^3 + 12x^2 + 22x - 4.$ (2) $-x^7 + 7x^6 + 3x^4 - 6x^3 - 2x - 11.$

(3) $\dfrac{4}{3}x^4 + 2x^2 - 1.$ (4) $-x^6 + x^5 + 35x^4 + 16x^3 + 84x^2 - 54x - 56.$

(5) $x^{13} - x^{11} + x^9 - x^7 + x^5 - x^3 + x.$

Q. Simplify:

(1) $6\left(x^7 + 7\,x^6 + 21\,x^5\right).$

(2) $-4\,\left(-\,x^2 + 5x + 3\right).$

(3) $$\frac{8x^{10} - 20x^8 + 24x^6 - 12x^4}{4}.$$

(4) $$\frac{1}{3}\,\left(\frac{3}{5}x^4 + \frac{3}{7}x^3 + \frac{3}{25}x^2 + \frac{3}{65}x\right).$$

(5) $3\,f\,(x)$ where

$$f\,(x) = x^5 + 4x^4 + 16x^3 + 22x^2 + 18x.$$

[**Answers**]:

(1) $6x^7 + 42x^6 + 126x^5.$

(2) $4x^2 - 20x - 12.$

(3) $2x^{10} - 5x^8 + 6x^6 - 3x^4.$

(4) $$\frac{1}{5}x^4 + \frac{1}{7}x^3 + \frac{1}{25}x^2 + \frac{1}{65}x.$$

(5) $3x^5 + 12x^4 + 48x^3 + 66x^2 + 54x.$