Math 105 TOPICS IN MATHEMATICS

STUDY GUIDE FOR MIDTERM EXAM – IC

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- §14. n-th root. Fractional exponents.
- Recall

$$\begin{aligned}
\sqrt[3]{0} &= 0, \\
\sqrt[3]{1} &= 1, \\
\sqrt[3]{8} &= 2, \\
\sqrt[3]{27} &= 3, \\
\sqrt[3]{64} &= 4, \\
\sqrt[3]{125} &= 5, \\
\sqrt[3]{216} &= 6, \\
\sqrt[3]{343} &= 7, \\
\sqrt[3]{512} &= 8, \qquad \text{and} \\
\sqrt[3]{729} &= 9.
\end{aligned}$$

Q. $\sqrt[3]{1000} = ?$ $\quad\sqrt[3]{1331} = ?$ $\quad\sqrt[3]{1728} = ?$ $\quad\sqrt[3]{4913} = ?$

Consult the table below, if necessary.

x	10	11	12	13	14	15	16	17	18	19
x^3	1000	1331	1728	2197	2744	3375	4096	4913	5832	6859

— $\sqrt[3]{1000} = 10.$ Indeed, $1000 = 10^3.$

$\sqrt[3]{1331} = 11.$ Indeed, $1331 = 11^3.$

$\sqrt[3]{1728} = 12.$ Indeed, $1728 = 12^3.$

$\sqrt[3]{4913} = 17.$ Indeed, $4913 = 17^3.$

So, in short:

If n is a non-negative integer, and if $\boxed{a = n^3}$ then $\boxed{\sqrt[3]{a} = n}$.

But the real issue here is,

$$\begin{array}{cccccc}
\sqrt[3]{2} =? & \sqrt[3]{3} =? & \sqrt[3]{4} =? & \sqrt[3]{5} =? & \sqrt[3]{6} =? & \sqrt[3]{7} =? \\
\sqrt[3]{9} =? & \sqrt[3]{10} =? & \sqrt[3]{11} =? & \sqrt[3]{12} =? & \sqrt[3]{13} =? & \sqrt[3]{14} =? \\
\sqrt[3]{15} =? & \sqrt[3]{16} =? & \sqrt[3]{17} =? & \sqrt[3]{18} =? & \sqrt[3]{19} =? & \sqrt[3]{20} =? \\
\sqrt[3]{21} =? & \sqrt[3]{22} =? & \sqrt[3]{23} =? & \sqrt[3]{24} =? & \sqrt[3]{25} =? & \sqrt[3]{26} =?
\end{array}$$

etc. (as you can see, I excluded $\sqrt[3]{0}$, $\sqrt[3]{1}$, $\sqrt[3]{8}$ and $\sqrt[3]{27}$).

Review. What is $\sqrt[3]{2}$?

$\sqrt[3]{2}$ is a number whose cube equals 2. Namely:

“ $\boxed{x = \sqrt[3]{2}}$ *is a number satisfying* $\boxed{x^3 = 2}$ ”.

Here, we ask the same question as last time: “Does such a number exist?” The answer is, yes, such a number indeed exists. This is just like last time we asserted that $\sqrt{2}$ exists. How do we find $\sqrt[3]{2}$? We can heuristically pull the decimal expression of $\sqrt[3]{2}$ as follows:

0. Observe

$$\begin{aligned} 1^3 &= 1, & \longleftarrow \text{ smaller than } 2 \\ 2^3 &= 8. & \longleftarrow \text{ bigger than } 2 \end{aligned}$$

So $\sqrt[3]{2}$ must sit between 1 and 2:

$$1 < \sqrt[3]{2} < 2.$$

1. Observe

$$\begin{aligned} 1.1^3 &= 1.331, & \\ 1.2^3 &= 1.728, & \longleftarrow \text{ smaller than } 2 \\ 1.3^3 &= 2.197, & \longleftarrow \text{ bigger than } 2 \end{aligned}$$

So $\sqrt[3]{2}$ must sit between 1.2 and 1.3:

$$1.2 < \sqrt[3]{2} < 1.3.$$

2. Observe

$$\begin{aligned} 1.21^3 &= 1.771561, & \\ 1.22^3 &= 1.815848, & \\ 1.23^3 &= 1.860867, & \\ 1.24^3 &= 1.906624, & \\ 1.25^3 &= 1.953125, & \longleftarrow \text{ smaller than } 2 \\ 1.26^3 &= 2.000376. & \longleftarrow \text{ bigger than } 2 \end{aligned}$$

So $\sqrt[3]{2}$ must sit between 1.25 and 1.26:

$$1.25 \;<\; \sqrt[3]{2} \;<\; 1.26.$$

3. Observe

$$\begin{aligned}
1.251^3 &= 1.957816251,\\
1.252^3 &= 1.962515008,\\
1.253^3 &= 1.967221277,\\
1.254^3 &= 1.971935064,\\
1.255^3 &= 1.976656375,\\
1.256^3 &= 1.981385216,\\
1.257^3 &= 1.986121593,\\
1.258^3 &= 1.990865512,\\
1.259^3 &= 1.995616979, && \longleftarrow \text{ smaller than } 2\\
1.260^3 &= 2.000376000. && \longleftarrow \text{ bigger than } 2
\end{aligned}$$

So $\sqrt[3]{2}$ must sit between 1.259 and 1.260:

$$1.259 \;<\; \sqrt[3]{2} \;<\; 1.260.$$

4. Observe

$$\begin{aligned}
1.2591^3 &= 1.996092541071,\\
1.2592^3 &= 1.996568178688,\\
1.2593^3 &= 1.997043891857,\\
1.2594^3 &= 1.997519680584,\\
1.2595^3 &= 1.997995544875,\\
1.2596^3 &= 1.998471484736,\\
1.2597^3 &= 1.998947500173,
\end{aligned}$$

$$
\begin{aligned}
1.2598^3 &= 1.999423591192, & & \\
1.2599^3 &= 1.999899757799, & \longleftarrow & \text{ smaller than } 2 \\
1.2600^3 &= 2.000376000000. & \longleftarrow & \text{ bigger than } 2
\end{aligned}
$$

So $\sqrt[3]{2}$ must sit between 1.2599 and 1.2600:

$$1.2599 \ < \ \sqrt[3]{2} \ < \ 1.2600.$$

So

$$\sqrt[3]{2} \ = \ 1.2599...$$

But of course this is endless. If you want to see more digits:

$$\sqrt[3]{2} \ = \ 1.25992104989487316476721060727822835057025146470 15... \ .$$

Most importantly, the decimal expression of $\sqrt[3]{2}$ continues forever, it never ends.

As for this, there is an algorithm for cube-rooting similar to the one for square-rooting which we have practiced in §12. But we choose not to discuss that.

The decimal expressions of $\sqrt[3]{3}$, $\sqrt[3]{4}$, $\sqrt[3]{5}$, $\sqrt[3]{6}$, and $\sqrt[3]{7}$ (up to the first fifty digits under the decimal point):

$$
\begin{aligned}
\sqrt[3]{3} &= 1.44224957030740838232163831078010958839186925349935... \ , \\
\sqrt[3]{4} &= 1.58740105196819947475170563927230826039149332789985... \ , \\
\sqrt[3]{5} &= 1.70997594667669698935310887254386010986805511054305... \ , \\
\sqrt[3]{6} &= 1.81712059283213965889121175632726050242821046314121... \ , \\
\sqrt[3]{7} &= 1.91293118277238910119911683954876028286243905034587... \ .
\end{aligned}
$$

Review. Nothing stops us from considering the fourth-root, fifth-root, and so on so forth.

$$\text{“}\ \boxed{x = \sqrt[4]{2}}\ \ \underline{\underline{\textit{is a number satisfying}}}\ \ \boxed{x^4 = 2}\ \text{”}.$$

$$\text{“}\ \boxed{x = \sqrt[5]{3}}\ \ \underline{\underline{\textit{is a number satisfying}}}\ \ \boxed{x^5 = 3}\ \text{”}.$$

More generally, for an arbitrary positive integer n, we may define the n-th root of a number.

Definition (n-th root).

Assume a is a positive number: $a > 0$. (Here, we do not assume that a is an integer. For example, a can be e.) Also, let n be a positive integer. Then

$$\text{“}\ \boxed{x = \sqrt[n]{a}}\ \ \underline{\underline{\textit{is a number satisfying}}}\ \ \boxed{x^n = a}\ \text{”}.$$

We call $\sqrt[n]{a}$ $\underline{\underline{\text{ the } n\text{-th root of } a\ .}}$

The square-root, the cube-root, the fourth-root, the fifth-root, *etc.* are called

“radicals.”

Also, the symbol $\sqrt[n]{}$ is called the radical symbol, or just the radical.

⋆ So, for $n = 2$, $\sqrt[n]{a}$ is $\sqrt[2]{a}$, and this is just the square-root of a. There is absolutely nothing wrong in writing the square-root of a as $\sqrt[2]{a}$, but it is customary that we allow ourselves to omit the tiny 2 in front of the radical symbol. So we usually write $\sqrt{a}$ for $\sqrt[2]{a}$.

Rule A. $$\boxed{\sqrt[n]{\sqrt[k]{a}} = \sqrt[nk]{a}}\,.$$

Q. Can we simplify

$$\sqrt{\sqrt{2}} \qquad ?$$

— Yes. $\sqrt[4]{2}$.

Q. Can we simplify

$$\sqrt[3]{\sqrt{2}} \qquad ?$$

— Yes. $\sqrt[6]{2}$.

Q. Can we simplify

$$\sqrt[4]{\sqrt[3]{2}} \qquad ?$$

— Yes. $\sqrt[12]{2}$.

Rule B. $$\boxed{\sqrt[nk]{a^n} = \sqrt[k]{a}}\,.$$

Q. Simplify $\sqrt[4]{4}$.

— $\sqrt{2}$.

Q. Simplify $\sqrt[6]{25}$.

— $\sqrt[3]{5}$.

Q. Simplify $\sqrt[15]{27}$.

— $\sqrt[5]{3}$.

Q. Use Rule A to simplify

(1) $\sqrt[3]{\sqrt{6}}$. (2) $\sqrt[4]{\sqrt[3]{10}}$.

— (1) $\sqrt[6]{6}$. (2) $\sqrt[12]{10}$.

Q. Use Rule B to simplify

(1) $\sqrt[6]{125}$. (2) $\sqrt[12]{343}$. (3) $\sqrt[16]{256}$.

(4) $\sqrt[12]{81}$.

— (1) $\sqrt{5}$. (2) $\sqrt[4]{7}$. (3) $\sqrt{2}$.

(4) $\sqrt[3]{3}$.

Rule C. $$\boxed{\left(\sqrt[n]{a}\right)\left(\sqrt[n]{b}\right) = \sqrt[n]{ab}}\,.$$

Rule C′. $$\boxed{\left(\sqrt[n]{a}\right)\left(\sqrt[n]{b}\right)\left(\sqrt[n]{c}\right) = \sqrt[n]{abc}}\,.$$

Rule C″. $$\boxed{\left(\sqrt[n]{a}\right)\left(\sqrt[n]{b}\right)\left(\sqrt[n]{c}\right)\left(\sqrt[n]{d}\right) = \sqrt[n]{abcd}}\,.$$

⋆ Rule C′, and Rule C″ follow from Rule C. To that extent we say that Rule C′ and Rule C″ are redundant.

Q. Are

$$\sqrt{2} \cdot \sqrt{3} \quad \text{and} \quad \sqrt{6}$$

the same, in view of the fact that $2 \cdot 3 = 6$?

— Yes, they are the same:

$$\sqrt{2} \cdot \sqrt{3} = \sqrt{6}.$$

Q. Are

$$\sqrt[3]{5} \cdot \sqrt[3]{7} \quad \text{and} \quad \sqrt[3]{35}$$

the same, in view of the fact that $5 \cdot 7 = 35$?

— Yes, they are the same:

$$\sqrt[3]{5} \cdot \sqrt[3]{7} = \sqrt[3]{35}.$$

Q. Use Rule C (or Rule C′, or Rule C″) to simplify

(1) $\sqrt{6} \cdot \sqrt{11}$. (2) $\sqrt[3]{3} \cdot \sqrt[3]{5}$. (3) $\sqrt[4]{2} \cdot \sqrt[4]{5} \cdot \sqrt[4]{13}$.

(4) $\sqrt[5]{2} \cdot \sqrt[5]{3} \cdot \sqrt[5]{4} \cdot \sqrt[5]{5}$.

— (1) $\sqrt{66}$. (2) $\sqrt[3]{15}$. (3) $\sqrt[4]{130}$. (4) $\sqrt[5]{120}$.

Rule D. $$\boxed{\left(\sqrt[k]{a}\right)^n = \sqrt[k]{a^n}}.$$

- §15. Fractional exponents.

Review. Adopting the following new notation is beneficial:

- **Alternative Notation.**

$$\boxed{\sqrt[n]{a} \;=\; a^{\frac{1}{n}}} \; .$$

Also, at this point we introduce

Definition. $\underline{\underline{\text{Define}}}$ $a^{\frac{n}{k}}$ $\underline{\underline{\text{as}}}$ $\left(a^n \right)^{\frac{1}{k}}$.

By virtue of Rule D from §14, this is equivalent to the following:

Definition. $\underline{\underline{\text{Define}}}$ $a^{\frac{n}{k}}$ $\underline{\underline{\text{as}}}$ $\left(a^{\frac{1}{k}} \right)^n$.

- This way we have defined

$$\boxed{a^r} \;\; \underline{\underline{\text{where}}} \;\; \boxed{r} \;\; \underline{\underline{\text{is a rational number.}}}$$

Now with this definition we can encaplsulate the above miscellaneous rules in a concise form:

Rule I. $\boxed{\left(ab \right)^r \;=\; a^r \, b^r}$.

Rule II. $\boxed{a^r \; a^s \;=\; a^{r+s}}$.

Rule III. $\boxed{\left(a^r \right)^s \;=\; a^{rs}}$.

Rule IV. $\boxed{a^0 \;=\; 1}$, $\boxed{1^r \;=\; 1}$.

$\star$ You might feel we should add the following to the above list

Rule I′. $$\Big(abc\Big)^r = a^r\, b^r\, c^r,$$

Rule I″. $$\Big(abcd\Big)^r = a^r\, b^r\, c^r\, d^r,$$

etc., but these are redundant.

As for why Rule II is true, let's look at the following easy special case first:

$$\begin{aligned} a^2 \cdot a^3 &= a\ a\ a\ a\ a \\ &= a^5\ , \end{aligned}$$

$$\begin{aligned} a^3 \cdot a^4 &= a\ a\ a\ a\ a\ a\ a \\ &= a^7\ , \end{aligned}$$

and so on. You can extrapolate and conclude

$$a^n\ a^\ell = a^{n+\ell} \qquad \Big(n,\ \ell :\ \text{positive } \underline{\underline{\text{integers}}}\Big).$$

Now, Rule II asserts

$$a^r\ a^s = a^{r+s} \qquad \Big(r,\ s :\ \text{positive } \underline{\underline{\text{rational numbers}}}\Big),$$

which is more general than the above. To see that the latter is indeed true, we first test it by some concrete example. Suppose

$$r = \frac{1}{4} \qquad \text{and} \qquad s = \frac{5}{6}.$$

then

$$r \;=\; \frac{1\,\cdot 3}{4\,\cdot 3} \;=\; \frac{3}{12} \qquad \text{and} \qquad r \;=\; \frac{5\,\cdot 2}{6\,\cdot 2} \;=\; \frac{10}{12}.$$

Accordingly,

$$\begin{aligned} r\,+\,s \quad &= \quad \frac{3}{12} + \frac{10}{12} \\ &= \quad \frac{3\,+\,10}{12} \quad = \quad \frac{13}{12}. \end{aligned}$$

Meanwhile,

$$\begin{aligned} a^r\; a^s \quad &= \quad a^{\frac{3}{12}}\; a^{\frac{10}{12}} \\ &= \quad \left(a^{\frac{1}{12}}\right)^3 \left(a^{\frac{1}{12}}\right)^{10}. \end{aligned}$$

At this point we treat $a^{\frac{1}{12}}$ as an individual number, call it b, so the above is

$$b^3\; b^{10} \quad = \quad b^{13}.$$

Remember that $b \;=\; a^{\frac{1}{12}}$, so this last outcome b^{13} becomes

$$\left(a^{\frac{1}{12}}\right)^{13}.$$

By definition, this equals

$$a^{\frac{13}{12}}.$$

To conclude, $a^r\;\, a^s \quad = \quad a^{r+s}$ is indeed true for $r \;=\; \dfrac{1}{4}$ and $s \;=\; \dfrac{5}{6}$.

It is now easy to take care of the case r and s are arbitrary positive rational numbers, by generalizing the above argument. Namely, write r and s as

$$r = \frac{n}{k}, \quad s = \frac{\ell}{k},$$

using appropriate positive integers n, ℓ and k, which is always feasible (common denominator technique, see "Supplement"). Then

$$\begin{aligned} a^r\, a^s &= a^{\frac{n}{k}}\, a^{\frac{\ell}{k}} \\ &= \left(a^{\frac{1}{k}}\right)^n \left(a^{\frac{1}{k}}\right)^\ell . \end{aligned}$$

At this point we may treat $a^{\frac{1}{k}}$ as an individual number, call it b, so the above is

$$b^n\, b^\ell.$$

Here n and ℓ are positive integers, thus we previously saw that this equals $b^{n+\ell}$. Now, remember that b equals $a^{\frac{1}{k}}$, so

$$\begin{aligned} b^{n+\ell} &= \left(a^{\frac{1}{k}}\right)^{n+\ell} \\ &= a^{\frac{n+\ell}{k}} \\ &= a^{\frac{n}{k}+\frac{\ell}{k}} . \end{aligned}$$

This is nothing else but a^{r+s}. In short, $a^r\, a^s = a^{r+s}$. This establishes Rule II.

⋆ Now, we may also highlight some variations of Rule II, such as

$$a^r\ a^s\ a^t \quad = \quad a^{r+s+t},$$

$$a^r\ a^s\ a^t\ a^u \quad = \quad a^{r+s+t+u},$$

etc., but these are redundant, just the same reason Rule I′, Rule I″, *etc.* are redundant. Indeed, these are immediate consequences of Rule II (apply Rule II repeatedly).

The above rules (Rules I–IV) are usually put together, and are referred to as the exponential laws. So let me highlight them one more time:

Exponential Laws. Let r and s be positive rational numbers. Let a and b be positive numbers (not necessarily rational numbers). Then

Rule I. $$\boxed{\left(ab\right)^r \quad = \quad a^r\, b^r}\ .$$

Rule II. $$\boxed{a^r\ a^s \quad = \quad a^{r+s}}\ .$$

Rule III. $$\boxed{\left(a^r\right)^s \quad = \quad a^{rs}}\ .$$

Rule IV. $$\boxed{a^0 \ = \ 1}\ , \qquad \boxed{1^r \ = \ 1}\ .$$

Q. Simplify

(1) $2^{\frac{2}{3}} \cdot \left(\frac{3}{2}\right)^{\frac{2}{3}}$. (2) $3^{\frac{1}{2}} \cdot 3^{\frac{5}{2}}$. (3) $\left(2^3\right)^{\frac{1}{4}}$.

(4) 100^0. (5) $1^{\frac{4}{7}}$.

— (1) $3^{\frac{2}{3}}$. (2) $3^3 = 27$. (3) $2^{\frac{3}{4}}$.
(4) 1. (5) 1.