Math 105 TOPICS IN MATHEMATICS

STUDY GUIDE FOR MIDTERM EXAM – IB

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- §8 continued.

- **Review — Fermat Primes, Mersenne Primes.**

Recall that 2-to-the-powers are

$$\begin{array}{rcr}
2^1 & = & 2, \\
2^2 & = & 4, \\
2^3 & = & 8, \\
2^4 & = & 16, \\
2^5 & = & 32, \\
2^6 & = & 64, \\
2^7 & = & 128, \\
2^8 & = & 256, \\
2^9 & = & 512, \\
2^{10} & = & 1024, \\
2^{11} & = & 2048, \\
2^{12} & = & 4096, \\
2^{13} & = & 8192, \\
2^{14} & = & 16384, \\
2^{15} & = & 32768, \\
2^{16} & = & 65536, \\
\cdot & & \cdot
\end{array}$$

While these seemingly have nothing to do with the prime numbers, adding 1 to, or subtracting 1 from, those suddenly have a bearing on the business of the mystery of primes.

Q. What is the definition of Fermat number ?

— A number of the form

$$\boxed{2^n + 1}, \qquad n\text{: a positive integer}$$

is called a Fermat number .

Q. What is the definition of a Fermat prime ?

— A Fermat number which is a prime number is called a Fermat prime .

Q. What is the definition of Mersenne number ?

— A number of the form

$$\boxed{2^n - 1}, \qquad n\text{: a positive integer}$$

is called a Mersenne umber .

Q. What is the definition of a Mersenne prime ?

— A Mersenne number which is a prime number is called a Mersenne prime .

Q. True or false: There are infinitely many prime numbers.

— True.

Q. Who proved it?

— Euclid.

Review continued. That does not mean that we $\big($humans on this planet$\big)$ have a list that contains infinitely many concrete examples of primes. In fact, no one has.

There is such a thing called '*the largest known prime*'.

Q. What does 'the largest known prime' exactly mean?

— Somebody has offered one particular number $\big($a positive integer$\big)$, a very very large number, and has mathematically proved that it is indeed a prime. Moreover, no one has offered another, larger, number and has mathematically proved that it is a prime.

Review continued. People are working on getting hold of larger and larger primes. So, the largetst known prime today may not be the largetst known prime tomorrow.

Q. True or false: For each given prime p, we, human being, do not know of a concrete formula for the next prime, or not even a formula that generates any prime larger than p.

— True.

Q. True or false: Somewhere out in the universe, there is a planet where there is an intelligence, something like us, humans, live there, and they do know such a formula.

— Who knows.

Review continued. Now, the largest known primes are 'typically' a Mersenne prime. That's one reason why Mersenne primes draw public attention. Meanwhile, Fermat primes, "kissing cousins" of Mersenne primes, are of special interest after the striking discovery by a mathematician named Gauss.

Q. What exactly is Gauss' discovery pertaining to Fermat primes?

— If N is a Fermat prime, then a regular polygon with N edges is drawn only using straightedge and compass. In particular, a regular heptadecagon $\big($= a regular 17-gon$\big)$ can be drawn only with straightedge and compass.

Q. Define 'regular n-gon'.

— Regular N-gon is a figure inscribed in a circle that has N straight edges and N vertices, and those N vertices are evenly distributed on the perimeter of the circle.

Review continued. Here, please don't dismiss it by saying "computers can draw just about any of those figures". There is a precise mathematical meaning attached to the expression "*something can be drawn only with straightedge and compass.*" When I talk about drawings of figures, what you are thinking is either an ink spead on a sheet or a collection of dots, or 'pixels', in case it is digitally drawn. But every stroke has thickness, no matter how thin it is, just that the thickness is thin enough so from a distance it looks like lines and circles but they are actually 'bands', and moreover the width of the bands is not exactly even, if you care to use a microscope to magnify it, partially because the surface of the paper $\big($or LCD screen$\big)$ is not exactly even or flat. But we draw figures in math classes, and our stance is we '*pretend*' that those strokes have no width.

In that sense, yes, of course, your computer can draw an '*approximate*' figure within the margin of the thickness of a stroke. But what I am talking about is something else. In mathematics, a statement "such and such polygon is drawn only with straightedge and compass" means that the pair of numbers that pinpoint the location of any of its vertices relative to the origin of the coordinate $\big($the coordinate readings of the referenced vertex$\big)$ both belong to a sequence of numbers where each member in that sequence arises as a root of a certain quadratic equation whose coefficients reside in a 'field generated by' the previous member of the same sequence, where a field generated by a certain number means the smallest number system that contains that number and all integers that is closed under addition, subtraction, multiplication, and division. So, in the context of feasibility of drawing figures, the computer's drawing ability is irrelevant.

Q. True or false: Certain regular polygons are mathematically proved to be impossible to be drawn only using straightedge and compass, comforming to the above definition. Also, if 'true', then give an example.

— True. Regular heptagons $\big(= 7\text{-gons}\big)$ cannot be drawn only using straightedge and compass.

Review continued. The following are open questions $\big($as in nobody has either proved or disproved them yet$\big)$:

(Open) Question 1. Are there infinitely many Fermat primes?

(Open) Question 2. Are there infinitely many Mersenne primes?

Q. In order for a Fermat number $2^n + 1$ to be a prime, n has to be what?

— n has to be a 2-to-the-power itself.

Q. Among the following Fermat numbers, which one$\big($s$\big)$ is/are Fermat prime$\big($s$\big)$?

$$\begin{aligned} F_1 &= 2^{2^1} + 1 = 2^2 + 1 = 5, \\ F_2 &= 2^{2^2} + 1 = 2^4 + 1 = 17, \\ F_3 &= 2^{2^3} + 1 = 2^8 + 1 = 257, \\ F_4 &= 2^{2^4} + 1 = 2^{16} + 1 = 65537, \qquad \text{and} \\ F_5 &= 2^{2^5} + 1 = 2^{32} + 1 = 4294967297. \end{aligned}$$

— F_1, F_2, F_3 and F_4 are Fermat primes. F_5 is not a Fermat prime, because

$$4294967297 = 641 \cdot 6700417.$$

Q. Who gave this factorization?

— Euler.

Review continued. Now, today with all the modern computer technology, all F_ks up to F_{42} are computed, and it is verified that none of them except the first four: F_1, F_2, F_3, and F_4, are primes. To this day no one knows if there is a Fermat prime other than F_1, F_2, F_3, and F_4.

Q. In order for a Mersenne number $2^n - 1$ to be a prime, n has to be what?

— n has to be a prime itself.

Q. True or false: If p is a prime, then $2^p - 1$ is a Mersenne prime. If false, then give a counterexample.

— False. Indeed, $2^{11} - 1 = 2047 = 23 \cdot 89.$

Review continued. This is why Question 2 "Are there infinitely many Mersenne primes?" makes sense. Here is the 'largest known prime', as of December, 2014, which happens to be a Mersenne prime:

$$2^{57885161} - 1.$$

This is a number that carries 17425170 digits.

• §9. Binomial expansions.

Review. Binomial Formula. Let n be a positive integer. Then

$$\boxed{\begin{aligned}&\left(x+y\right)^n\\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \binom{n}{3} x^{n-3}y^3 + \cdots\\ &\quad + \binom{n}{n-3} x^3\,y^{n-3} + \binom{n}{n-2} x^2\,y^{n-2} + \binom{n}{n-1} x\,y^{n-1} + \binom{n}{n} y^n.\end{aligned}}$$

For $n = 1, 2, 3, 4, 5$ and 6, this is

(1) $\left(x+y\right)^1 = \binom{1}{0} x + \binom{1}{1} y,$

(2) $\left(x+y\right)^2 = \binom{2}{0} x^2 + \binom{2}{1} x\,y + \binom{2}{2} y^2,$

(3) $\left(x+y\right)^3 = \binom{3}{0} x^3 + \binom{3}{1} x^2\,y + \binom{3}{2} x\,y^2 + \binom{3}{3} y^3,$

(4) $\left(x+y\right)^4 = \binom{4}{0} x^4 + \binom{4}{1} x^3\,y + \binom{4}{2} x^2\,y^2 + \binom{4}{3} x\,y^3 + \binom{4}{4} y^4,$

(5) $\left(x+y\right)^5$

$$= \binom{5}{0} x^5 + \binom{5}{1} x^4\,y + \binom{5}{2} x^3\,y^2 + \binom{5}{3} x^2\,y^3 + \binom{5}{4} x\,y^4 + \binom{5}{5} y^5,$$

(6) $\left(x+y\right)^6$

$$= \binom{6}{0} x^6 + \binom{6}{1} x^5\,y + \binom{6}{2} x^4\,y^2 + \binom{6}{3} x^3\,y^3 + \binom{6}{4} x^2\,y^4$$
$$+ \binom{6}{5} x\,y^5 + \binom{6}{6} y^6.$$

Q. Throw actual numbers in the binomial coefficients in the above (1)–(6).

— (1) $\left(x + y\right)^1 = x + y,$

(2) $\left(x + y\right)^2 = x^2 + \underline{\underline{2}}\, x\, y + y^2,$

(3) $\left(x + y\right)^3 = x^3 + \underline{\underline{3}}\, x^2 y + \underline{\underline{3}}\, x\, y^2 + y^3,$

(4) $\left(x + y\right)^4 = x^4 + \underline{\underline{4}}\, x^3 y + \underline{\underline{6}}\, x^2 y^2 + \underline{\underline{4}}\, x\, y^3 + y^4,$

(5) $\left(x + y\right)^5 = x^5 + \underline{\underline{5}}\, x^4 y + \underline{\underline{10}}\, x^3 y^2 + \underline{\underline{10}}\, x^2 y^3 + \underline{\underline{5}}\, x\, y^4 + y^5,$

(6) $\left(x + y\right)^6 = x^6 + \underline{\underline{6}}\, x^5 y + \underline{\underline{15}}\, x^4 y^2 + \underline{\underline{20}}\, x^3 y^3 + \underline{\underline{15}}\, x^2 y^4 + \underline{\underline{6}}\, x\, y^5 + y^6,$

Q. Substitute $y = 1$ in the above (1)–(6).

— (1) $\left(x + 1\right)^1 = x + 1,$

(2) $\left(x + 1\right)^2 = x^2 + 2\,x + 1,$

(3) $\left(x + 1\right)^3 = x^3 + 3\,x^2 + 3\,x + 1,$

(4) $\left(x + 1\right)^4 = x^4 + 4\,x^3 + 6\,x^2 + 4\,x + 1,$

(5) $\left(x + 1\right)^5 = x^5 + 5\,x^4 + 10\,x^3 + 10\,x^2 + 5\,x + 1,$

(6) $\left(x + 1\right)^6 = x^6 + 6\,x^5 + 15\,x^4 + 20\,x^3 + 15\,x^2 + 6\,x + 1.$

Q. Substitute $y = 2$ in the above (1)–(6).

— (1) $\left(x + 2\right)^1 = x + 2,$

(2) $\left(x + 2\right)^2 = x^2 + 2 \cdot x \cdot 2 + 2^2$

$= x^2 + 4x + 4,$

(3) $\left(x + 2\right)^3 = x^3 + 3 \cdot x^2 \cdot 2 + 3 \cdot x \cdot 2^2 + 2^3$

$= x^3 + 6x^2 + 12x + 8,$

(4) $\left(x + 2\right)^4 = x^4 + 4 \cdot x^3 \cdot 2 + 6 \cdot x^2 \cdot 2^2 + 4 \cdot x \cdot 2^3 + 2^4$

$= x^4 + 8x^3 + 24x^2 + 32x + 16,$

(5) $\left(x + 2\right)^5 = x^5 + 5 \cdot x^4 \cdot 2 + 10 \cdot x^3 \cdot 2^2 + 10 \cdot x^2 \cdot 2^3 + 5 \cdot x \cdot 2^4 + 2^5$

$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32,$

(6) $\left(x + 2\right)^6 = x^6 + 6 \cdot x^4 \cdot 2 + 15 \cdot x^4 \cdot 2^2 + 20 \cdot x^3 \cdot 2^3 + 15 \cdot x^2 \cdot 2^4$

$+ 6 \cdot x \cdot 2^5 + 2^6$

$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64.$

Q. Substitute $y = -1$ in the above (1)–(6).

— (1) $\left(x - 1\right)^{1} = x - 1,$

(2) $\left(x - 1\right)^{2} = x^2 - 2x + 1,$

(3) $\left(x - 1\right)^{3} = x^3 - 3x^2 + 3x - 1,$

(4) $\left(x - 1\right)^{4} = x^4 - 4x^3 + 6x^2 - 4x + 1,$

(5) $\left(x - 1\right)^{5} = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1,$

(6) $\left(x - 1\right)^{6} = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1.$

Q. Substitute $y = -2$ in the above (1)–(6).

— (1) $\left(x - 2\right)^{1} = x - 2,$

(2) $\left(x - 2\right)^{2} = x^2 - 4x + 4,$

(3) $\left(x - 2\right)^{3} = x^3 - 6x^2 + 12x - 8,$

(4) $\left(x - 2\right)^{4} = x^4 - 8x^3 + 24x^2 - 32x + 16,$

(5) $\left(x - 2\right)^{5} = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32,$

(6) $\left(x - 2\right)^{6} = x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64.$

Q. Expand each of

$\left(x+7\right)^2.$ $\left(x+6\right)^3.$ $\left(x+3\right)^4.$ $\left(x+1\right)^6.$

$\left(x-2\right)^2.$ $\left(x-4\right)^4.$ $\left(x-3\right)^5.$ $\left(x-1\right)^7.$

— $$\left(x+7\right)^2 = x^2 + 14x + 49.$$

$$\left(x+6\right)^3 = x^3 + 18x^2 + 108x + 216.$$

$$\left(x+3\right)^4 = x^4 + 12x^3 + 54x^2 + 108x + 81.$$

$$\left(x+1\right)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1.$$

$$\left(x-2\right)^2 = x^2 - 4x + 4.$$

$$\left(x-4\right)^4 = x^4 - 16x^3 + 96x^2 - 256x + 256.$$

$$\left(x-3\right)^5 = x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243.$$

$$\left(x-1\right)^7 = x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1.$$

- §10. e. Intro.

Review. "Not for nothing", let's "compare"

(1) $$\left(1+\frac{1}{1}\right)^1,$$

(2) $$\left(1+\frac{1}{2}\right)^2,$$

(3) $$\left(1+\frac{1}{3}\right)^3,$$

(4) $$\left(1+\frac{1}{4}\right)^4,$$

(5) $$\left(1+\frac{1}{5}\right)^5,$$

$$\vdots$$

In decimals, these are

(1) $$2\ ,$$

(2) $$1.5\ \cdot\ 1.5\quad ,$$

(3) $$\big(1.333333...\big)\ \cdot\ \big(1.333333...\big)\ \cdot\ \big(1.333333...\big)\quad ,$$

(4) $$1.25\ \cdot\ 1.25\ \cdot\ 1.25\ \cdot\ 1.25\quad ,\qquad \text{and}$$

(5) $$1.2\ \cdot\ 1.2\ \cdot\ 1.2\ \cdot\ 1.2\ \cdot\ 1.2\quad .$$

Which one is the smallest? Which one is the largest? No calculators. Indeed, in the same context you will soon encounter something that will ultimately make you think twice about an indiscriminate use of calculators. Now, recall

$$
\begin{aligned}
10^2 &= 10 \cdot 10 = 100 \quad \big(\text{one hundred}\big). \\
10^3 &= 10 \cdot 10 \cdot 10 = 1000 \quad \big(\text{one thousand}\big). \\
10^4 &= 10 \cdot 10 \cdot 10 \cdot 10 = 10000 \quad \big(\text{ten thousand}\big). \\
10^5 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100000 \quad \big(\text{one hundred thousand}\big). \\
10^6 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1000000 \quad \big(\text{one million}\big). \\
10^7 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000000 \quad \big(\text{ten million}\big). \\
10^8 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100000000 \quad \big(\text{one hundred million}\big). \\
10^9 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1000000000 \quad \big(\text{one billion}\big). \\
10^{10} &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000000000 \\
&\qquad \big(\text{ten billion}\big). \\
10^{11} &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100000000000 \\
&\qquad \big(\text{one hundred billion}\big). \\
10^{12} &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1000000000000 \\
&\qquad \big(\text{one trillion}\big). \\
\vdots & \qquad \vdots \qquad \ddots
\end{aligned}
$$

- More generally, for a positive integer n,

$$
\boxed{10^n = 1\,\underbrace{0\,0\,0\,0\,0\,\ldots\,0}_{n}} \,.
$$

Now, trusting the calculator to do the following will be ill-advised:

$$(10^3) \qquad \left(1+\frac{1}{10^3}\right)^{10^3},$$

$$(10^6) \qquad \left(1+\frac{1}{10^6}\right)^{10^6},$$

$$(10^9) \qquad \left(1+\frac{1}{10^9}\right)^{10^9},$$

$$(10^{12}) \qquad \left(1+\frac{1}{10^{12}}\right)^{10^{12}},$$

$$(10^{15}) \qquad \left(1+\frac{1}{10^{15}}\right)^{10^{15}},$$

$$(10^{18}) \qquad \left(1+\frac{1}{10^{18}}\right)^{10^{18}},$$

$$\vdots$$

Indeed, one of our designated calculator models gives

(10^3)	2.7169..,
(10^6)	2.7182804..,
(10^9)	2.718281827..,
(10^{12})	2.718281828..,
(10^{15})	1,
(10^{18})	1,

(In other models too '1' starts to show up, though where exactly depends.)

The answers for part (10^{15}) and part (10^{18}) are inaccurate $\big($way off the mark$\big)$. Part (10^{15}) should be a tiny bit bigger than part (10^{12}); part (10^{18}) should be a tiny tiny bit bigger than part (10^{15}), and so on. And this trend continues for forever. Your calculator rounds numbers where it shouldn't, and sometimes that leads to an error. Below is an absolutely irrefutable logic to validate my claim. For the sake of illustration, let's dissect part (5):

$$
\begin{aligned}
(5) \qquad \left(1+\frac{1}{5}\right)^5 \quad &= \quad \binom{5}{0} \cdot 1^5 \\
&\quad + \binom{5}{1} \cdot 1^4 \cdot \left(\frac{1}{5}\right) \\
&\quad + \binom{5}{2} \cdot 1^3 \cdot \left(\frac{1}{5}\right)^2 \\
&\quad + \binom{5}{3} \cdot 1^2 \cdot \left(\frac{1}{5}\right)^3 \\
&\quad + \binom{5}{4} \cdot 1 \cdot \left(\frac{1}{5}\right)^4 \\
&\quad + \binom{5}{5} \cdot \left(\frac{1}{5}\right)^5 \\
&= \quad 1 \\
&\quad + \frac{1}{1} \cdot \frac{5}{5} \\
&\quad + \frac{1}{1\cdot 2} \cdot \frac{5}{5} \cdot \frac{4}{5} \\
&\quad + \frac{1}{1\cdot 2\cdot 3} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \\
&\quad + \frac{1}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \\
&\quad + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5} .
\end{aligned}
$$

In short,part (5) equals

$$1 \tag{term 5-0}$$

$$+ \frac{1}{1} \cdot \frac{5}{5} \tag{term 5-1}$$

$$+ \frac{1}{1 \cdot 2} \cdot \frac{5}{5} \cdot \frac{4}{5} \tag{term 5-2}$$

$$+ \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \tag{term 5-3}$$

$$+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \tag{term 5-4}$$

$$+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5} . \tag{term 5-5}$$

Similarly, part (6) equals

$$1 \tag{term 6-0}$$

$$+ \frac{1}{1} \cdot \frac{6}{6} \tag{term 6-1}$$

$$+ \frac{1}{1 \cdot 2} \cdot \frac{6}{6} \cdot \frac{5}{6} \tag{term 6-2}$$

$$+ \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \tag{term 6-3}$$

$$+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \tag{term 6-4}$$

$$+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \tag{term 6-5}$$

$$+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6} . \tag{term 6-6}$$

Now, I contend

$$
\begin{aligned}
(\text{term 5-0}) &= (\text{term 6-0}),\\
(\text{term 5-1}) &= (\text{term 6-1}),\\
(\text{term 5-2}) &< (\text{term 6-2}),\\
(\text{term 5-3}) &< (\text{term 6-3}),\\
(\text{term 5-4}) &< (\text{term 6-4}),\\
(\text{term 5-5}) &< (\text{term 6-5}).
\end{aligned}
$$

Indeed, these are just

$$
\begin{aligned}
1 &= 1\,,\\
\frac{1}{1}\cdot\frac{5}{5} &= \frac{1}{1}\cdot\frac{6}{6},\\
\frac{1}{1\cdot 2}\cdot\frac{5}{5}\cdot\frac{4}{5} &< \frac{1}{1\cdot 2}\cdot\frac{6}{6}\cdot\frac{5}{6},\\
\frac{1}{1\cdot 2\cdot 3}\cdot\frac{5}{5}\cdot\frac{4}{5}\cdot\frac{3}{5} &< \frac{1}{1\cdot 2\cdot 3}\cdot\frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6},\\
\frac{1}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{5}{5}\cdot\frac{4}{5}\cdot\frac{3}{5}\cdot\frac{2}{5} &< \frac{1}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6}\cdot\frac{3}{6}\,,\\
\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{5}{5}\cdot\frac{4}{5}\cdot\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{1}{5} &< \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6}\cdot\frac{3}{6}\cdot\frac{2}{6},
\end{aligned}
$$

which are all true. In the above, (term 6-6) was not involved, but we can see that (term 6-6) is bigger than 0. So, in sum:

$$
\begin{array}{rcccl}
 & (\text{term } 5\text{-}0) & = & (\text{term } 6\text{-}0) & \\
 & (\text{term } 5\text{-}1) & = & (\text{term } 6\text{-}1) & \\
 & (\text{term } 5\text{-}2) & < & (\text{term } 6\text{-}2) & \\
 & (\text{term } 5\text{-}3) & < & (\text{term } 6\text{-}3) & \\
 & (\text{term } 5\text{-}4) & < & (\text{term } 6\text{-}4) & \\
 & (\text{term } 5\text{-}5) & < & (\text{term } 6\text{-}5) & \\
+) & 0 & < & (\text{term } 6\text{-}6) & \\
\hline
 & \text{part } (5) & < & \text{part } (6) &
\end{array}
$$

So, to conclude, part (6) is bigger than part (5):

(5) $$\left(1+\frac{1}{5}\right)^5, \qquad \text{and}$$

(6) $$\left(1+\frac{1}{6}\right)^6,$$

To write this as an inequality:

$$\left(1+\frac{1}{5}\right)^5 \quad < \quad \left(1+\frac{1}{6}\right)^6.$$

I didn't use a calculator.

If you employ the same logic as above, use Binomial Formula, then you will arrive at the conclusion part (7) is bigger than part (6); part (8) is bigger than part (7), and so on. More generally, part $(n+1)$ is bigger than part (n).

- So, the calculator did wrong. Mathematical computer software is more reliable, and it gives us

(10^3)	$2.716923932235892457383088121947577 1...$
(10^6)	$\underline{\underline{2.71}}828046931937688381979970845435 63...$
(10^9)	$\underline{\underline{2.71828}}1827099904322376644023860 3328...$
(10^{12})	$\underline{\underline{2.71828182}}8457686094446059194614 1537...$
(10^{15})	$\underline{\underline{2.71828182845}}9043876219373241831 2906...$
(10^{18})	$\underline{\underline{2.71828182845904}}5234001146557123 1398...$
(10^{21})	$\underline{\underline{2.71828182845904523}}5358928330438 4329...$
(10^{24})	$\underline{\underline{2.7182818284590452353}}60286112211 7482...$
(10^{27})	$\underline{\underline{2.71828182845904523536028}}7469993 5215...$
(10^{30})	$\underline{\underline{2.7182818284590452353602874}}71351 3033...$

The list indicates that the figures will not grow arbitrarily large, but will get stagnant. Indeed, one can theoretically prove

Claim. The numbers

$$\left(1+\frac{1}{n}\right)^n, \tag{n}$$

$n = 1, 2, 3, 4, 5, \cdots$, cannot become arbitrarily large. Indeed, the digit before the decimal point in the decimal expression of each of these numbers is 2. In other words, these numbers are all less than 3.

- §11. *e*. Continued.

Review. There is a way to give a mathematical reasoning why the above claim is true. But first a real life example. Before everything, in the following 'Metaphor', we relax the smallest currency unit, meaning: In reality, we cannot divide one cent. But here we work on a model where one can divide any dollar amount by any large number (integer). Also, we never round figures. So, one-third of a dollar is never the same as 33 cents (because 33 cents is one-third of 99 cents).

Metaphor. Now, you open a bank account, deposit a dollar in that account. Your bank offers 10 percent interest annually. After one year, your balance is a dollar and ten cents. But suppose another bank offers 100 percent interest annually. Then you probably want to forget the first bank and rush to the second bank, right? So you actually went to the second bank, with 100 percent annual interest rate. There you opened a bank account, and deposited a dollar in that account. Then after one year your balance is two dollars. A much better deal.

Intermission. As a matter of fact, with any interest rate the gist of what I am going to show you is the same. The difference is a constant multiplication in the exponent. Having or not having that constant is mathematically insubstantial. With the 100 percent rate we can make that constant 1. The general case is a simple tweak of it. So let's stick with the second bank with 100 percent interest.

Metaphor resumed. Now, in the second bank, with 100 percent interest, you never withdraw money, or make additional deposit. You just let your money sit there. After the first year, the balance is two dollars, of which one dollar is accrued as an interest. After the second year, should the balance be how much?

If you say three dollars, then that's correct, if there is no compounding interest. If you say four dollars, then that's correct too, the interest is compounded.

So, both are correct. From now on we only focus on compound interest. So, suppose the second bank offers a compound interest with 100 percent interest rate annually. But then there is a third bank, a competitor, that advertises as follows: They offer the same interest rate of 100 percent annually, but they calculate the intest more frequently than once a year, namely, twice a year.

Don't be misled: What the third bank is *not* saying is they offer 100 percent interest semi-anunally, so your money would grow like after six months what was originally a dollar would grow into two dollars, and then after another six months that two dollars would further grow into four dollars, and so on. That's *not* what they are advertising. Rather, they keep the 100 percent annual interest rate, but the 100 percent annual rate translates to 50 percent semi-annual interest rate. But if compounding takes place semi-annually with that rate, that's a better deal than the 100 percent annual interest rate with compounding taking place just annually. Are you following me? Let's mathematically dissect.

- With the second bank $\big($annual interest rate is 100 percent, compounded annually$\big)$, after 1 year your balance is

$$\$\Big(1+1\Big).$$

- With the third bank $\big($annual interest rate is 100 percent, compounded semi-annually$\big)$, after $\dfrac{1}{2}$ year your balance is

$$\$\left(1+\frac{1}{2}\right),$$

  and after 1 year it is

$$\$\left(1+\frac{1}{2}\right)^2.$$

Now, there is a fourth bank, that tries to outplay the third bank, and they advertise the 100 percent annual interest rate, which itself is the same, but they compound the interest three times a year, each time applying $\dfrac{1}{3}$ of 100 percent rate. Then

- With the fourth bank, your balance after $\dfrac{1}{3}$ year is

$$\$\left(1+\frac{1}{3}\right),$$

after $\dfrac{2}{3}$ year it is

$$\$\left(1+\frac{1}{3}\right)^2,$$

and after 1 year it is

$$\$\left(1+\frac{1}{3}\right)^3.$$

Now, there is a fifth bank, that tries to outplay the fourth bank, and they advertise the 100 percent annual interest rate, which itself is the same, but they compound the interest quarter-annually $\big($four times a year$\big)$, each time applies $\dfrac{1}{4}$ of 100 percent interest rate. Then

- With the fifth bank, your balance after $\dfrac{1}{4}$ year is

$$\$\left(1+\frac{1}{4}\right),$$

after $\dfrac{2}{4}$ year it is

$$\$\left(1+\frac{1}{4}\right)^2,$$

after $\dfrac{3}{4}$ year it is

$$\$\left(1+\frac{1}{4}\right)^3,$$

and after 1 year it is

$$\$\left(1+\frac{1}{4}\right)^4.$$

And so on so forth. At this point, let's just focus on the balance after one year, in each scenario $\big($with each of the second through the fifth banks$\big)$. With a dollar of a deposit, with 100 percent annual interest rate, and the compounding takes place n times a year, with $n = 1, 2, 3$ and 4, the balance is

$$\$\left(1+1\right)^{1},$$

$$\$\left(1+\frac{1}{2}\right)^{2},$$

$$\$\left(1+\frac{1}{3}\right)^{3},$$

and

$$\$\left(1+\frac{1}{4}\right)^{4},$$

respectively. We have mathematically verified that this is an increasing sequence. In other words, the higher the compounding frequency is, the more additional benefit you receive. But here, the real question is, as we increase the compounding frequency, does the benefit increase unlimitedly?

The answer is 'no'. The additional gain will ultimately diminish as you keep increasing the compounding frequency. This is exactly the claim we are making $\big($at the end of §9 and at the beginning of §10$\big)$. Below is how we mathematically verify the claim.

Recall that

$$\left(1+\frac{1}{5}\right)^{5}$$

equals the following quantity:

$$
\begin{aligned}
&1 \\
&+ \frac{1}{1} \cdot \underbrace{\frac{5}{5}}_{\substack{\| \\ ①}} \\
&+ \frac{1}{1 \cdot 2} \cdot \underbrace{\frac{5}{5} \cdot \frac{4}{5}}_{\substack{\| \\ ②}} \\
&+ \frac{1}{1 \cdot 2 \cdot 3} \cdot \underbrace{\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5}}_{\substack{\| \\ ③}} \\
&+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \underbrace{\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5}}_{\substack{\| \\ ④}} \\
&+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \underbrace{\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5}}_{\substack{\| \\ ⑤}}.
\end{aligned}
$$

But if you look at the portion underlined, they are all less than 1, except ① equals 1:

$$\frac{5}{5} = 1,$$

$$\frac{5}{5} \cdot \frac{4}{5} < 1,$$

$$\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} < 1,$$

$$\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \ < \ 1,$$

$$\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5} \ < \ 1$$

$\big($see "Review of Lectures – X Supplement"$\big)$. So, if you replace the underlined parts with 1, then the resulting quantity becomes bigger $\big($once again, see "Review of Lectures – X Supplement"$\big)$. In short,

$$\left(1+\frac{1}{5}\right)^5$$

$$< \quad 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}.$$

Now, if you further compare this latter quantity with

$$1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 2} + \frac{1}{1\cdot 2\cdot 2\cdot 2} + \frac{1}{1\cdot 2\cdot 2\cdot 2\cdot 2}$$

$$= \quad 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4},$$

then this last quantity $1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}$ is clearly bigger. Now, we know the fact that this last quantity $1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}$ is $\frac{1}{2^4}$ short of 3.

So, in short,

$$\left(1+\frac{1}{5}\right)^5 \quad < \quad 3.$$

The same argument works for

$$(n) \qquad \left(1+\frac{1}{n}\right)^n,$$

with any n. In sum, we draw the following conclusion:

Conclusion. For an arbitrary positive integer $n = 1, 2, 3, 4, \cdots$,

$$\boxed{\left(1+\frac{1}{n}\right)^n \quad < \quad 3.}$$

• The next question is to identify the 'limit' of these numbers. Namely, we are going to figure out the balance after one year with "continuous" compounding, namely, the frequency of compounding n approaches to infinity.

- §12. Factorials. Definition of e.

Review. Recall

$$\begin{aligned}
1! &= 1,\\
2! &= 2\cdot 1,\\
3! &= 3\cdot 2\cdot 1,\\
4! &= 4\cdot 3\cdot 2\cdot 1,\\
5! &= 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
6! &= 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
7! &= 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
8! &= 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
9! &= 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
10! &= 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
&\vdots \qquad\qquad\qquad\qquad \ddots
\end{aligned}$$

Pronunciation:

$1!$ = "one factorial",
$2!$ = "two factorial",
$3!$ = "three factorial",
$4!$ = "four factorial",
$5!$ = "five factorial",
$6!$ = "six factorial",
$7!$ = "seven factorial",
$8!$ = "eight factorial",
$9!$ = "nine factorial".
$10!$ = "ten factorial".

Q. How to pronounce $20!$? Can you spell that out?

—
$$\begin{aligned} 20! \quad = \quad & 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \\ & \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \, . \end{aligned}$$

The actual figure:

$$\begin{aligned} 20! \quad = \quad & 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \\ & \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ = \quad & 2432902008176640000. \end{aligned}$$

This one you can do by hand.

So, as you can imagine, the sequence of factorial numbers grow very rapidly. To use a real life example:

Q. You started a lemonade business. Words spread fast and the sales dramatically increased (Table 1 below):

day	$
1	1
2	2
3	6
4	24
5	120
6	720
⋮	⋮

On Day n, your sales will be n times the previous day sales. Suppose the same patterns hold until Day 50. Then the sales on Day 50 is how much?

— The dollar amount of the sales on Day n is $n!$ ("n factorial"). So the dollar amount of the sales on Day 50 is exactly $50!$ ("fifty factorial"):

$$\begin{aligned} 50! \quad = \quad & 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43 \cdot 42 \cdot 41 \\ & \cdot 40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31 \\ & \cdot 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \\ & \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \\ & \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \,. \end{aligned}$$

Review. The actual figure of this number is as follows (computer aided):

$$50! = 30414093201713378043612608166064768844377641568960512000000000000.$$

This is a 64-digit number.

Q. Is it the case that the numbers like $1000000!$, and $1000000000!$, your computer can spit out the exact answers?

— No, because the number of digits are astronomical, so it is not possible to display on the screen the entire answer. Your computer still provides ball-park figures.

Q. How do computer calculate those ballpark figures of large number factorials?

— The moment it recognizes the problem, it immediately says to itself "forget brute-force calculation", to multiply 1 through one million (one billion, ...). But then it quickly identifies the theorem pre-installed that helps computing the figure with the least amount of calculations.

Review. That theorem itself is actually old, was devised around 1730. So, there is actually some "theory behind computing factorials". We will cover this in the second half of the semester. That has to do with the coverage of the rest of this section: the behavior of

$$\left(1+\frac{1}{n}\right)^n; \qquad n = 1, 2, 3, 4, 5, \cdots$$

as n grows larger. We already know two things about this sequence. (a) This is an increasing sequence. (b) No matter how large n is, the number cannot exceed 3. Here, we are not abruptly changing the subject, from factorials to somethign else. The following simple observation is actually very helpful toward our goal:

> " *When you write a binomial coefficient in a fraction form, the denominator is a factorial number.* "

Example. $\displaystyle \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = \frac{7 \cdot 6 \cdot 5}{3!}.$

Q. Write out each of $\binom{9}{5}$, and $\binom{10}{4}$ in a similar fashion.

—
$$\binom{9}{5} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{5!},$$

$$\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!}.$$

A general formula is

- **Binomial coefficients expressed in terms of factorials — I.**

Let n and k be integers, with $0 < k < n$. Then the binomial coefficient $\binom{n}{k}$ is written as

$$\boxed{\binom{n}{k} = \frac{n \, (n-1) \, (n-2) \, \cdots \, (n-k+1)}{k\,!}}\,.$$

Now, let's incorporate this perspective with what we did earlier. With the factorial symbol, what we have worked out in §10 is paraphrased as

(5) $\left(1+\dfrac{1}{5}\right)^5$

$$
\begin{aligned}
= \quad & 1 && \text{(term 5-0)}\\
+ \; & \frac{1}{1!} \cdot \frac{5}{5} && \text{(term 5-1)}\\
+ \; & \frac{1}{2!} \cdot \frac{5}{5} \cdot \frac{4}{5} && \text{(term 5-2)}\\
+ \; & \frac{1}{3!} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} && \text{(term 5-3)}\\
+ \; & \frac{1}{4!} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} && \text{(term 5-4)}\\
+ \; & \frac{1}{5!} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5} \,. && \text{(term 5-5)}
\end{aligned}
$$

What's clear is

Fact A-5. $$\left(1+\frac{1}{5}\right)^5 \quad < \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}.$$

Similarly,

(6) $$\left(1+\frac{1}{6}\right)^6$$

$$= \quad 1 \qquad \text{(term 6-0)}$$

$$+ \frac{1}{1!} \cdot \frac{6}{6} \qquad \text{(term 6-1)}$$

$$+ \frac{1}{2!} \cdot \frac{6}{6} \cdot \frac{5}{6} \qquad \text{(term 6-2)}$$

$$+ \frac{1}{3!} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \qquad \text{(term 6-3)}$$

$$+ \frac{1}{4!} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \qquad \text{(term 6-4)}$$

$$+ \frac{1}{5!} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \qquad \text{(term 6-5)}$$

$$+ \frac{1}{6!} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6} \qquad \text{(term 6-6)}$$

What's clear is

Fact A-6. $$\left(1+\frac{1}{6}\right)^6 \quad < \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}.$$

On the other hand,

(7) $$\left(1+\frac{1}{7}\right)^7$$

$$
\begin{aligned}
= \quad & 1 && \text{(term 7-0)}\\
+ \quad & \frac{1}{1!} \cdot \frac{7}{7} && \text{(term 7-1)}\\
+ \quad & \frac{1}{2!} \cdot \frac{7}{7} \cdot \frac{6}{7} && \text{(term 7-2)}\\
+ \quad & \frac{1}{3!} \cdot \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} && \text{(term 7-3)}\\
+ \quad & \frac{1}{4!} \cdot \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} \cdot \frac{4}{7} && \text{(term 7-4)}\\
+ \quad & \frac{1}{5!} \cdot \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} \cdot \frac{4}{7} \cdot \frac{3}{7} && \text{(term 7-5)}\\
+ \quad & \frac{1}{6!} \cdot \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} \cdot \frac{4}{7} \cdot \frac{3}{7} \cdot \frac{2}{7} && \text{(term 7-6)}\\
+ \quad & \bigl(\text{an extra term, which is positive}\bigr),
\end{aligned}
$$

(8) $$\left(1+\frac{1}{8}\right)^8$$

$$
\begin{aligned}
= \quad & 1 && \text{(term 8-0)}\\
+ \quad & \frac{1}{1!} \cdot \frac{8}{8} && \text{(term 8-1)}\\
+ \quad & \frac{1}{2!} \cdot \frac{8}{8} \cdot \frac{7}{8} && \text{(term 8-2)}\\
+ \quad & \frac{1}{3!} \cdot \frac{8}{8} \cdot \frac{7}{8} \cdot \frac{6}{8} && \text{(term 8-3)}\\
+ \quad & \frac{1}{4!} \cdot \frac{8}{8} \cdot \frac{7}{8} \cdot \frac{6}{8} \cdot \frac{5}{8} && \text{(term 8-4)}\\
+ \quad & \frac{1}{5!} \cdot \frac{8}{8} \cdot \frac{7}{8} \cdot \frac{6}{8} \cdot \frac{5}{8} \cdot \frac{4}{8} && \text{(term 8-5)}\\
+ \quad & \frac{1}{6!} \cdot \frac{8}{8} \cdot \frac{7}{8} \cdot \frac{6}{8} \cdot \frac{5}{8} \cdot \frac{4}{8} \cdot \frac{3}{8} && \text{(term 8-6)}\\
+ \quad & \bigl(\text{extra terms, which are all positive}\bigr),
\end{aligned}
$$

and so on. We have observed how each of

(term 5-2), (term 6-2), (term 7-2), (term 8-2), ... ;

(term 5-3), (term 6-3), (term 7-3), (term 8-3), ... ;

(term 5-4), (term 6-4), (term 7-4), (term 8-4), ... , and

(term 5-5), (term 6-5), (term 7-5), (term 8-5), ... ,

grow, and concluded:

Conclusion. As n grows larger, the sum of

- the deficit of by how much (term n-2) is short of $\frac{1}{2!}$,
- the deficit of by how much (term n-3) is short of $\frac{1}{3!}$,
- the deficit of by how much (term n-4) is short of $\frac{1}{4!}$, and
- the deficit of by how much (term n-5) is short of $\frac{1}{5!}$,

will get closer and closer to 0, whereas (term n-6) is at least

$$\frac{1}{6!} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6}$$

(which is just (term 6-6)) and it keeps growing. So ultimately, at some point, when n becomes large enough, the number $\left(1+\frac{1}{n}\right)^n$ exceeds the value

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}.$$

So we arrive at the conclusion:

Fact B-5. If you choose a large enough n, then

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}.$$

- Extrapolation leads:

Fact B-6. If you choose a large enough n, then

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}.$$

Fact B-7. If you choose a large enough n, then

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!}.$$

Fact B-8. If you choose a large enough n, then

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}.$$

and so on.

We can acutally compress all of thse into one single statement, which is as follows:

Fact B. Let k be an arbitrarily chosen positive integer, and fixed. By choosing a large enough n, we can make the following inequality true:

$$\boxed{\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}} \ .$$

$\star$ Meanwhile, we can also extrapolate our prior observation and conclude:

Fact A. Let k be an arbitrary positive integer. Then

$$\boxed{\left(1+\frac{1}{k}\right)^k \quad < \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}} \ .$$

So, what do these mean altogether? Yes, Fact A and Fact B mean precisely as follows:

Fact A and Fact B Compiled in one.

" *If you compare* $\boxed{\left(1+\frac{1}{k}\right)^k}$ and

$$\boxed{1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}} \ ,$$

for the same k, then always the latter is bigger, however, the latter

is exceeded by $\boxed{\left(1+\frac{1}{n}\right)^n}$ for a different, larger, n.

This way we arrive at the following definition:

Definition 1. <u>The limit</u>

$$\boxed{\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n}$$

<u>means the 'threshold' number, the smallest number which</u> $\left(1+\frac{1}{n}\right)^n$ <u>cannot</u>

<u>exceed when n runs through the entire positive integers.</u>

Definition 2. <u>The limit</u>

$$\boxed{\lim_{k \to \infty} \left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{k!}\right)}$$

<u>means the 'threshold' number, the smallest number which</u>

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{k!}$$

<u>cannot exceed when k runs through the entire positive integers.</u>

Conclusion. <u>The above two limits are indeed equal.</u>

$$\boxed{\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = \lim_{k \to \infty} \left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{k!}\right)}\,.$$

Definiiton 3 $\Big($**The precise mathematical definition of** $\boldsymbol{e}\Big)$**.**

$$\boxed{\begin{aligned} e &= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \\ &= \lim_{k \to \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}\right). \end{aligned}}$$

- **Decimal expression of** $\boldsymbol{e}$**.**

$$e = 2.71828182845904523536028747135266249775724709369 99...$$

$$\begin{aligned}
1 + \frac{1}{1!} &= 2, \\
1 + \frac{1}{1!} + \frac{1}{2!} &= 2.5, \\
1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} &= 2.6666666... \ , \\
1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} &= 2.7083333... \ , \\
1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} &= 2.7166666... \ , \\
1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} &= 2.7180555... \ , \\
1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} &= 2.7182539... \ , \\
\vdots \qquad & \qquad \vdots
\end{aligned}$$

If you want to see more digits: The values

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{k!}$$

for $k = 1, 2, 3, 4, \cdots, 30$ (up to the 30th digit under the decimal point):

$$\begin{array}{lcl}
k = 1 & \Longrightarrow & 2 \\
k = 2 & \Longrightarrow & 2.5 \\
k = 3 & \Longrightarrow & 2.666666666666666666666666666666... \\
k = 4 & \Longrightarrow & 2.708333333333333333333333333333... \\
k = 5 & \Longrightarrow & 2.716666666666666666666666666666... \\
k = 6 & \Longrightarrow & 2.718055555555555555555555555555... \\
k = 7 & \Longrightarrow & 2.718253968253968253968253968253... \\
k = 8 & \Longrightarrow & 2.718278769841269841269841269841... \\
k = 9 & \Longrightarrow & 2.718281525573192239858906525573... \\
k = 10 & \Longrightarrow & 2.718281801146384479717813051146... \\
k = 11 & \Longrightarrow & 2.718281826198492865159531826198... \\
k = 12 & \Longrightarrow & 2.718281828286168563946341724119... \\
k = 13 & \Longrightarrow & 2.718281828446759002314557870113... \\
k = 14 & \Longrightarrow & 2.718281828458229747912287594827... \\
k = 15 & \Longrightarrow & 2.718281828458994464285469576474... \\
k = 16 & \Longrightarrow & 2.718281828459042259058793450327... \\
k = 17 & \Longrightarrow & 2.718281828459045070516047795848... \\
k = 18 & \Longrightarrow & 2.718281828459045226708117481710... \\
k = 19 & \Longrightarrow & 2.718281828459045234928752728335... \\
k = 20 & \Longrightarrow & 2.718281828459045235339784490666... \\
k = 21 & \Longrightarrow & 2.718281828459045235359357431729... \\
k = 22 & \Longrightarrow & 2.718281828459045235360247110869... \\
k = 23 & \Longrightarrow & 2.718281828459045235360285792570... \\
k = 24 & \Longrightarrow & 2.718281828459045235360287404308... \\
k = 25 & \Longrightarrow & 2.718281828459045235360287468777...
\end{array}$$

$$k = 26 \implies 2.718281828459045235360287471257...$$
$$k = 27 \implies 2.718281828459045235360287471349...$$
$$k = 28 \implies 2.718281828459045235360287471352...$$
$$k = 29 \implies 2.718281828459045235360287471352...$$
$$k = 30 \implies 2.718281828459045235360287471352...$$

Q. Once again, can you recite two definitions of e?

—

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n,$$

$$e = \lim_{k \to \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}\right).$$

- §13. Square roots.

Review. Recall

$$\begin{aligned} \sqrt{0} &= 0, \\ \sqrt{1} &= 1, \\ \sqrt{4} &= 2, \\ \sqrt{9} &= 3, \\ \sqrt{16} &= 4, \\ \sqrt{25} &= 5, \\ \sqrt{36} &= 6, \\ \sqrt{49} &= 7, \\ \sqrt{64} &= 8, \quad \text{and} \\ \sqrt{81} &= 9. \end{aligned}$$

Q. $\sqrt{100} = ?$ $\quad \sqrt{144} = ?$ $\quad \sqrt{225} = ?$ $\quad \sqrt{324} = ?$

Consult the table below, if necessary.

x	10	11	12	13	14	15	16	17	18	19
x^2	100	121	144	169	196	225	256	289	324	361

—

$$\begin{aligned} \sqrt{100} &= 10. \quad \text{Indeed,} \quad 100 = 10^2. \\ \sqrt{144} &= 12. \quad \text{Indeed,} \quad 144 = 12^2. \\ \sqrt{225} &= 15. \quad \text{Indeed,} \quad 225 = 15^2. \\ \sqrt{324} &= 18. \quad \text{Indeed,} \quad 324 = 18^2. \end{aligned}$$

So, in short:

$$\underline{\underline{\textit{If } n \textit{ is a non-negative integer, and if}}} \quad \boxed{a = n^2} \quad \underline{\underline{\text{then}}} \quad \boxed{\sqrt{a} = n} \ .$$

But the real issue here is,

$$\begin{array}{llllll}
\sqrt{2} =? & \sqrt{3} =? & \sqrt{5} =? & \sqrt{6} =? & \sqrt{7} =? & \sqrt{8} =? \\
\sqrt{10} =? & \sqrt{11} =? & \sqrt{12} =? & \sqrt{13} =? & \sqrt{14} =? & \sqrt{15} =? \\
\sqrt{17} =? & \sqrt{18} =? & \sqrt{19} =? & \sqrt{20} =? & \sqrt{21} =? & \sqrt{22} =? \\
\sqrt{23} =? & \sqrt{24} =? & \sqrt{26} =? & \sqrt{27} =? & \sqrt{28} =? & \sqrt{29} =?
\end{array}$$

etc. (as you can see, I excluded $\sqrt{0}$, $\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$ and $\sqrt{25}$).

Review. What is $\sqrt{2}$?

$\sqrt{2}$ is a number whose square equals 2. Namely:

$$\text{“} \boxed{x = \sqrt{2}} \quad \underline{\underline{\textit{is a number satisfying}}} \quad \boxed{x^2 = 2} \text{”.}$$

But do we know such a number? Does such a number exist? Today's discussion focuses on whether such a number really exists. The answer is, yes, such a number really exists. How do we find it? We can heuristically pull the decimal expression of $\sqrt{2}$ as follows:

0. Observe

$$\begin{aligned} 1^2 &= 1, & \longleftarrow \ \text{smaller than } 2 \\ 2^2 &= 4. & \longleftarrow \ \text{bigger than } 2 \end{aligned}$$

So $\sqrt{2}$ must sit between 1 and 2:

$$1 < \sqrt{2} < 2.$$

1. Observe

$$\begin{aligned}
1.1^2 &= 1.21, \\
1.2^2 &= 1.44, \\
1.3^2 &= 1.69, \\
1.4^2 &= 1.96, && \longleftarrow \text{ smaller than } 2 \\
1.5^2 &= 2.25, && \longleftarrow \text{ bigger than } 2
\end{aligned}$$

So $\sqrt{2}$ must sit between 1.4 and 1.5:

$$1.4 < \sqrt{2} < 1.5.$$

2. Observe

$$\begin{aligned}
1.41^2 &= 1.9881, && \longleftarrow \text{ smaller than } 2 \\
1.42^2 &= 2.0164, && \longleftarrow \text{ bigger than } 2
\end{aligned}$$

So $\sqrt{2}$ must sit between 1.41 and 1.42:

$$1.41 < \sqrt{2} < 1.42.$$

3. Observe

$$\begin{aligned}
1.411^2 &= 1.990921, \\
1.412^2 &= 1.993744, \\
1.413^2 &= 1.996569, \\
1.414^2 &= 1.999396, && \longleftarrow \text{ smaller than } 2 \\
1.415^2 &= 2.002225. && \longleftarrow \text{ bigger than } 2
\end{aligned}$$

So $\sqrt{2}$ must sit between 1.414 and 1.415:

$$1.414 < \sqrt{2} < 1.415.$$

4. Observe

$$\begin{aligned} 1.4141^2 &= 1.99967881, \\ 1.4142^2 &= 1.99996164, && \longleftarrow \text{ smaller than } 2 \\ 1.4143^2 &= 2.00024449. && \longleftarrow \text{ bigger than } 2 \end{aligned}$$

So $\sqrt{2}$ must sit between 1.4142 and 1.4143:

$$1.4142 < \sqrt{2} < 1.4143.$$

So

$$\sqrt{2} = 1.4142... \ .$$

But of course this is endless. If you want to see more digits:

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769... \ .$$

Most importantly, the decimal expression of $\sqrt{2}$ continues forever, it never ends.

As for this, there is a more efficient algorithm.

$$
\begin{array}{ccccccccccl}
\boxed{1}\,. & & \boxed{4} & & \boxed{1} & & \boxed{4} & & \boxed{2}\ \ldots & & \\
\cline{1-9}
\sqrt{\;}\ 2\,. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
\boxed{1} & & & & & & & & & \longleftarrow & \text{(line 0)} \\
\cline{1-3}
\boxed{1} & \boxed{0} & \boxed{0} & & & & & & & & \\
 & \boxed{9} & \boxed{6} & & & & & & & \longleftarrow & \text{(line 1)} \\
\cline{1-5}
 & & \boxed{4} & \boxed{0} & \boxed{0} & & & & & & \\
 & & \boxed{2} & \boxed{8} & \boxed{1} & & & & & \longleftarrow & \text{(line 2)} \\
\cline{3-7}
 & & \boxed{1} & \boxed{1} & \boxed{9} & \boxed{0} & \boxed{0} & & & & \\
 & & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{9} & \boxed{6} & & & \longleftarrow & \text{(line 3)} \\
\cline{3-9}
 & & & & \boxed{6} & \boxed{0} & \boxed{4} & \boxed{0} & \boxed{0} & & \\
 & & & & \boxed{5} & \boxed{6} & \boxed{5} & \boxed{6} & \boxed{4} & \longleftarrow & \text{(line 4)} \\
\cline{5-9}
 & & & & & \boxed{3} & \boxed{8} & \boxed{3} & \boxed{6} & & \\
 & & & & & & \vdots & & & \ddots &
\end{array}
$$

Let's dissect.

- Start with

$$\begin{array}{ll}
& \boxed{a}\,.\quad \boxed{}\quad \boxed{}\quad \boxed{}\quad \boxed{}\;\cdots \\
\sqrt{} & 2\,.\;0\;0\quad 0\;0\quad 0\;0\quad 0\;0 \\
& \boxed{} \qquad\qquad\qquad\qquad \longleftarrow \quad \text{(line 0)}
\end{array}$$

- You see a on top. a is one of 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. We are going to decide a.

- Choose the largest a such that a^2 does not exceed 2. So $a = 1$. Register your answer $a = 1$ on top. At the same time, place $a^2 = 1$ in (line 0) as indicated. Subtract (line 0) from the line right above it:

$$\begin{array}{ll}
& \boxed{1}\,.\quad \boxed{b}\quad \boxed{}\quad \boxed{}\quad \boxed{}\;\cdots \\
\sqrt{} & 2\,.\;0\;0\quad 0\;0\quad 0\;0\quad 0\;0 \\
& \boxed{1} \\
\hline
& \boxed{1}\;\boxed{0}\;\boxed{0} \\
& \boxed{}\;\boxed{}\;\boxed{} \qquad\qquad\qquad \longleftarrow \quad \text{(line 1)}
\end{array}$$

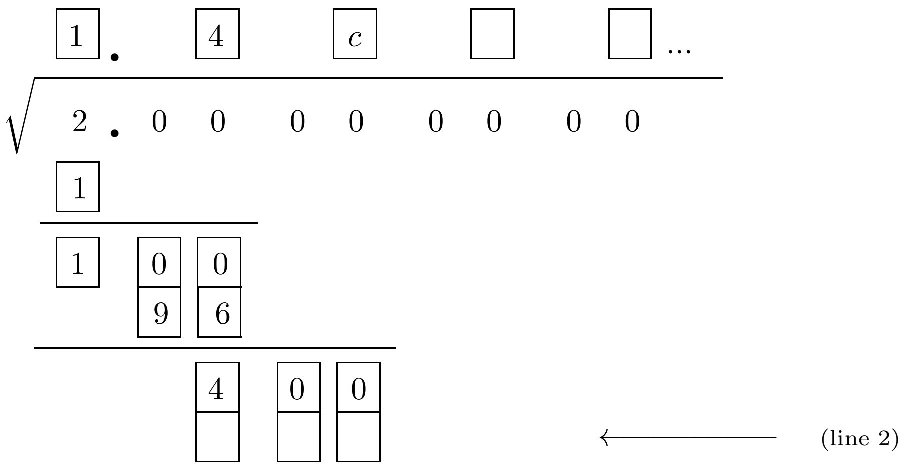
- Now the subtraction was performed. 00 was dragged down from the top. At this point you see 100 right above (line 1).

- Now you see b on top. b is one of 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. We are going to decide b.

- Choose the largest b such that

$$20 \cdot \boxed{1} \cdot b \quad + \quad b^2$$

does not exceed 100, where $\boxed{1}$ is in the left of $\boxed{b}$. So $b = 4$. Register your answer $b = 4$ on top. At the same time, place $20 \cdot 1 \cdot b + b^2 = 96$ in (line 1) as indicated. Subtract (line 1) from the line right above it:



- Now the subtraction was performed. 00 was dragged down from the top. At this point you see 400 right above (line 2).

- Now you see c on top. c is one of 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. We are going to decide c.

- Choose the largest c such that

$$20 \cdot \boxed{14} \cdot c \quad + \quad c^2$$

does not exceed 400, where $\boxed{14}$ is in the left of $\boxed{c}$. So $c = 1$. Register your answer $c = 1$ on top. At the same time, place $20 \cdot 14 \cdot c + c^2 = 281$ in (line 2) as indicated. Subtract (line 2) from the line right above it:

$$
\begin{array}{rrrrrl}
\boxed{1}\,. & \boxed{4} & \boxed{1} & \boxed{d} & \boxed{}\ \cdots & \\
\hline
\sqrt{\;2\,.} & 0\;0 & 0\;0 & 0\;0 & 0\;0 & \\
\boxed{1} & & & & & \\
\hline
\boxed{1} & \boxed{0}\,\boxed{0} & & & & \\
 & \boxed{9}\,\boxed{6} & & & & \\
\hline
 & \boxed{4} & \boxed{0}\,\boxed{0} & & & \\
 & \boxed{2} & \boxed{8}\,\boxed{1} & & & \\
\hline
 & \boxed{1} & \boxed{1}\,\boxed{9} & \boxed{0}\,\boxed{0} & & \\
 & \boxed{} & \boxed{}\,\boxed{} & \boxed{}\,\boxed{} & \longleftarrow & \text{(line 3)}
\end{array}
$$

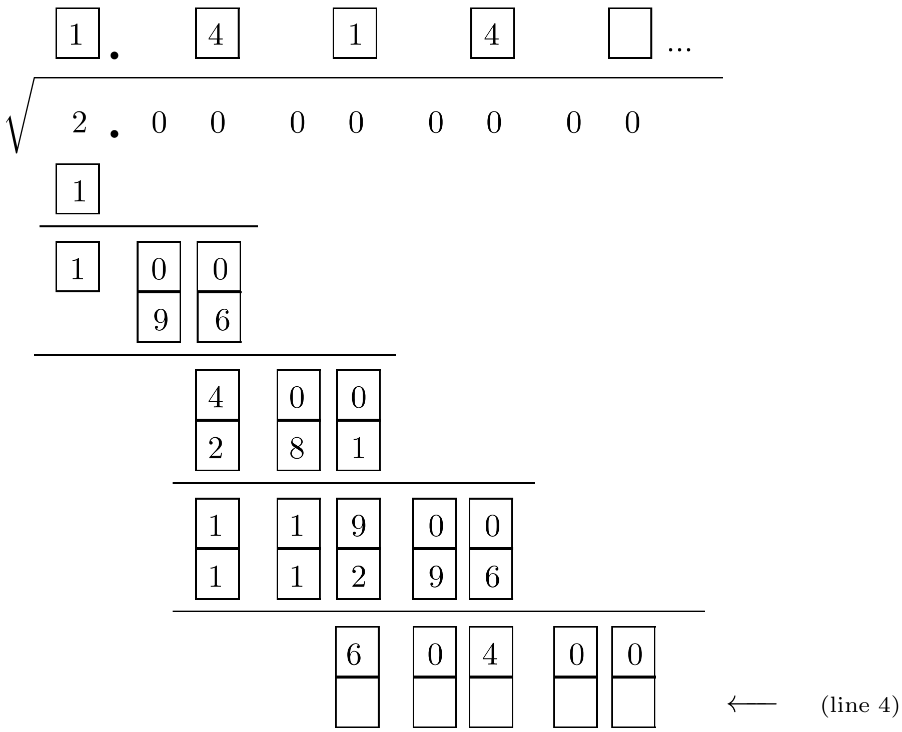
- Now the subtraction was performed. 00 was dragged down from the top. At this point you see 11900 right above (line 3).

- Now you see d on top. d is one of 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. We are going to decide d.

- Choose the largest d such that

$$20 \cdot \boxed{141} \cdot d \quad + \quad d^2$$

does not exceed 11900, where $\boxed{141}$ is in the left of $\boxed{d}$. So $d = 4$. Register your answer $d = 4$ on top. At the same time, place $20 \cdot 141 \cdot d + d^2 = 11296$ in (line 3) as indicated. Subtract (line 3) from the line right above it:



And so on so forth.

You can continue this procedure, and get as many digits under the decimal point as you want for the number $\sqrt{2}$. The computation becomes harder as you move on, though. Indeed, in this method, the size of the number you have to deal with (in terms of how many digits it carries) grows proportionately to the number of all the past steps.

Q. Do the same for $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{e}$, up to the fourth place under the decimal point.

— $\sqrt{3} \;=\; 1.7320...$

[**Work**]:

$$
\begin{array}{l}
\boxed{1}\,.\;\;\boxed{7}\;\;\boxed{3}\;\;\boxed{2}\;\;\boxed{0}\;\ldots \\
\sqrt{3\,.\,00\;\;00\;\;00\;\;00} \\
\boxed{1} \\
\hline
\boxed{2}\;\boxed{0}\,\boxed{0} \\
\boxed{1}\;\boxed{8}\,\boxed{9} \\
\hline
\quad\boxed{1}\,\boxed{1}\;\boxed{0}\,\boxed{0} \\
\quad\boxed{1}\,\boxed{0}\;\boxed{2}\,\boxed{9} \\
\hline
\qquad\boxed{7}\,\boxed{1}\;\boxed{0}\,\boxed{0} \\
\qquad\boxed{6}\,\boxed{9}\;\boxed{2}\,\boxed{4} \\
\hline
\qquad\quad\boxed{1}\;\boxed{7}\,\boxed{6}\;\boxed{0}\,\boxed{0} \\
\qquad\quad\boxed{}\;\boxed{}\,\boxed{}\;\boxed{}\,\boxed{0} \\
\hline
\qquad\quad\boxed{1}\;\boxed{7}\,\boxed{6}\;\boxed{0}\,\boxed{0} \\
\qquad\qquad\quad\vdots\qquad\qquad\ddots
\end{array}
$$

—

$$\sqrt{5} \quad = \quad 2.2360\ldots$$

[**Work**]:

$$
\begin{array}{llllllllllll}
 & \boxed{2}\,. & & \boxed{2} & & & \boxed{3} & & & \boxed{6} & & \boxed{0}\ \ldots \\
\hline
\sqrt{\ } & 5\,. & 0 & 0 & & 0 & 0 & & 0 & 0 & 0 & 0 \\
 & \boxed{4} & & & & & & & & & & \\
\cline{2-4}
 & \boxed{1} & \boxed{0} & \boxed{0} & & & & & & & & \\
 & & \boxed{8} & \boxed{4} & & & & & & & & \\
\cline{2-6}
 & & \boxed{1} & \boxed{6} & \boxed{0} & \boxed{0} & & & & & & \\
 & & \boxed{1} & \boxed{3} & \boxed{2} & \boxed{9} & & & & & & \\
\cline{3-8}
 & & & \boxed{2} & \boxed{7} & \boxed{1} & \boxed{0} & \boxed{0} & & & & \\
 & & & \boxed{2} & \boxed{6} & \boxed{7} & \boxed{9} & \boxed{6} & & & & \\
\cline{3-11}
 & & & & & \boxed{3} & \boxed{0} & \boxed{4} & \boxed{0} & \boxed{0} & & \\
 & & & & & \boxed{\ } & \boxed{\ } & \boxed{\ } & \boxed{\ } & \boxed{0} & & \\
\cline{6-11}
 & & & & & \boxed{3} & \boxed{0} & \boxed{4} & \boxed{0} & \boxed{0} & & \\
 & & & & & & & \vdots & & & \ddots &
\end{array}
$$

— $\sqrt{e} \;=\; 1.6487...$

[**Work**]:

$$
\begin{array}{cccccc}
1\,. & 6 & 4 & 8 & 7 & \ldots \\
\hline
2\,. & 7\;1 & 8\;2 & 8\;1 & 8\;2 & \\
1 & & & & & \\
\cline{1-2}
1 & 7\;1 & & & & \\
1 & 5\;6 & & & & \\
\hline
 & 1\;5 & 8\;2 & & & \\
 & 1\;2 & 9\;6 & & & \\
\cline{2-4}
 & 2 & 8\;6 & 8\;1 & & \\
 & 2 & 6\;3 & 0\;4 & & \\
\cline{2-5}
 & & 2\;3 & 7\;7 & 8\;2 & \\
 & & 2\;3 & 0\;7 & 6\;9 & \\
\cline{3-5}
 & & & 7\;0 & 1\;3 & \\
 & & & \vdots & & \ddots
\end{array}
$$

Q. Recall the definition of a rational number.

— A number of the form $\dfrac{k}{m}$ where k and m are integers and m is not equal to 0 is called a rational number.

Q. Recall the definition of an irrational number.

— A number which is not a rational number is called an irrational number.

Q. Is $\sqrt{2}$ is rational, or irrational?

— $\sqrt{2}$ is irrational.

Q. Who proved it?

— Euclid.

Q. Which one of the following are rational numbers?

$$\frac{1}{2}, \quad \frac{2}{3}, \quad 5, \quad \frac{7}{3}, \quad -2, \quad 0, \quad \frac{3}{10}, \quad -\frac{11}{6}, \quad -1000.$$

— All of those.

Review. Agree that there is an alternative definition of a rational number:

Alternative Definition (Rational numbers).

A rational number is a number which falls into either (i) or (ii):

(i) its decimal expression stops after finitely many digits under the decimal point (this includes an integer), or

(ii) its decimal expression contains a portion ('unit') made of a finite number of consecutive digits, and the whole decimal expression of that number is an infinite times iteration of that unit, except possibly a finite number of leading digits.

Q. Convert each of the following decimally expressed numbers into an integer divided by an integer form.

0.52, 0.3125.

— $0.52 = \dfrac{13}{25}$, and $0.3125 = \dfrac{5}{16}$

Q. Convert each iof the following fractions into a decimal expression.

$$\frac{1}{275}, \qquad \frac{764}{70}.$$

$$\frac{1}{275} = 0.0036363636... \qquad \frac{764}{70} = 10.9142857142857142857....$$

Q. Which one of the following is a rational number?

$$\sqrt{2}, \quad \sqrt{3}, \quad \sqrt{4}, \quad \sqrt{5}, \quad \sqrt{6}, \quad \sqrt{7}, \quad \sqrt{8}, \quad \sqrt{9}, \quad \sqrt{10}.$$

— $\sqrt{4} = 2$ and $\sqrt{9} = 3$ are rational, others are irrational.

Q. Prove that $\sqrt{2}$ is irrational.

⋆ The method of proof below is called <u>proof by contradiction</u>.

Proof. Suppose $\sqrt{2}$ is written as

$$\sqrt{2} = \frac{k}{m}$$

using some integers k and m $\big($where $m \neq 0\big)$.

First, if both k and m are even, then we may simultaneously divide both the numerator and the denominator by 2 $\big($and the value of the fraction stays the same$\big)$. After that procedure, suppose both the numerator and the denominator still remain to be even, then we repeat the same procedure as many times as necessary until at least one of the numerator and the denominator becomes odd. Thus we may assume, without loss of generality, that at least one of k and m is odd.

Under this assumption, square the both sides of the identity $\sqrt{2} = \dfrac{k}{m}$, thus

$$2 = \frac{k^2}{m^2}.$$

This is the same as

$$2\,m^2 = k^2.$$

The left-hand side of this last identity is clearly even, so this last identity forces its right-hand side to be also even. That in turn implies k is even, because if k is odd then k^2 is odd. But then k being even implies k^2 is divisible by 4. So by virtue of the above last identity $2\,m^2$ is divisible by 4, or the same to say, m^2 is divisible by 2, or the same to say, m^2 is even. This implies that m is even. In short, both k and m are even. This contradicts our assumption, that at least one of k and m is odd. The proof is complete. $\square$