Math 105 TOPICS IN MATHEMATICS

STUDY GUIDE FOR MIDTERM EXAM – IA

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• §1. What is mathematics? — The Riemann Hypothesis.

Q. What is a short description of the term 'conjecture' in math?

— It is an unsolved problem. A mathematical statement believed to be true but no one has proved or disproved it.

Q. Give an example of an outstanding conjecture.

— 'The Riemann Hypothesis'.

Review. Bernhard Riemann (1826-1866), has come up with this conjecture. After having encountered a very curious and peculiar phenomenon while investigating what's known today as Riemann's zeta function, Riemann saw that this is very likely true. So he decided to submit this as a conjecture. No one before him had come up with the same. And that was 1859. Little did he know was it would quickly become known as an absolutely impenetrable problem. In fact, It stood for the next 155+ years, had refuted the relentless ultra-intense scrutiny by the top notch mathematical geniuses in the subsequent eras, aided by the most advanced cutting-edge math technology devised specifically by those people to assault it. It still remains as an open problem today, March 6th, 2015. Anybody who solves this problem today would instantly become famous. An organization called Clay Mathematical Institute (CMI) offers some prize money on this problem along with a half-dozen other math problems. The amount is one million dollars.

Q. Briefly, why is the Riemann Hypothesis so important?

— Because this has to do with the so-called 'distribution of prime numbers'.

Q. What is a prime number? Give its precise definition.

— A positive integer which cannot be written as ab with two smaller positive integers a and b.

Q. How many prime numbers are there?

— Infinitely many.

Q. Identify all the prime numbers between 2 and 50 (circle all prime numbers among the numbers below).

	2,	3,	4,	5,	6,	7,	8,	9,	10,
11,	12,	13,	14,	15,	16,	17,	18,	19,	20,
21,	22,	23,	24,	25,	26,	27,	28,	29,	30.
31,	32,	33,	34,	35,	36,	37,	38,	39,	40.
41,	42,	43,	44,	45,	46,	47,	48,	49,	50.

— 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

Q. The only even prime number is 2. Can you explain why?

— First, 2 itself is a prime number because if $2 = ab$ and a and b are positive integers then clearly either a or b has to equal 2. Other even numbers are multiples of 2, so clearly they are not prime.

Q. How many prime numbers are there?

— Infinitely many.

Q. Who proved it?

— Euclid.

Q. True or false: There are 1000000 (one million) consecutive integers none of which is a prime.

— True.

Q. True or false: There are 10^{100} consecutive integers none of which is a prime.

— True.

Q. True or false: No matter how large a number, call it N, you choose, there are N consecutive integers none of which is a prime.

— True.

Q. True or false: No matter how large a number, call it N, you choose, there are two primes both of which are above N and their gap is less than 70000000 $\big($seventy million$\big)$.

— True.

Q. Who proved it?

— Yitang Zhang.

Review. That was in 2013. Before Yitang Zhang, no one has proved that such a bound exists. After Zhang's result, this gap of 70000000 was trimmed down to 246. The conjectured optimal gap is 2 $\big($the twin prime conjecture$\big)$.

Q. Briefly explain why computers cannot solve these problems.

— These problems clearly all deal with infinity. What your computer can tell you is whether up to a certain large number your working hyopthesis is true.

Q. Give an example of an outstanding problem that has been solved by computers.

— Four color problem.

Q. Why was it feasible to solve it by computer?

— Because the problem was essentially breakable into finite algorithms $\big($steps$\big)$.

Q. Who proved that there are indeed infinitely many s at which the Riemann zeta function takes the zero value on the so-called 'critical line'?

— G. H. Hardy.

- §2. Sum of consecutive integers.

Review. Analyze the patterns in the answers of the following:

(1) $1 \;=?$

(2) $1 + 2 \;=?$

(3) $1 + 2 + 3 \;=?$

(4) $1 + 2 + 3 + 4 \;=?$

(5) $1 + 2 + 3 + 4 + 5 \;=?$

(6) $1 + 2 + 3 + 4 + 5 + 6 \;=?$

(7) $1 + 2 + 3 + 4 + 5 + 6 + 7 \;=?$

(8) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \;=?$

(9) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \;=?$

(10) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \;=?$

$\vdots$

— The heuristic way to figure out the answers:

Line (10):

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

11

11

11

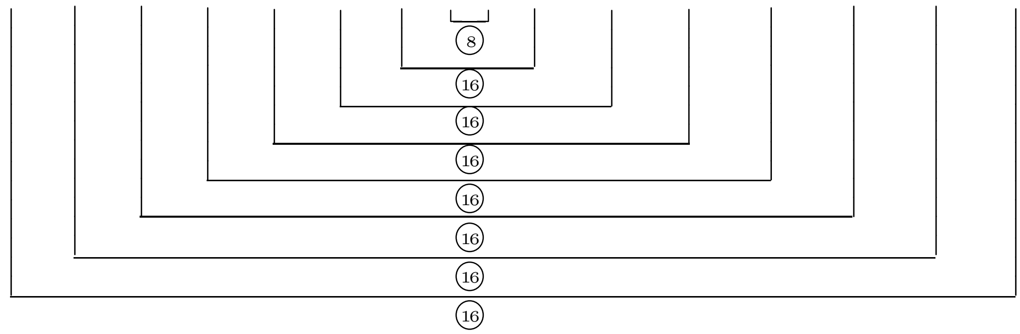
11

11

$$= \; 11 \cdot 5 \; = \; 55.$$

Line (15):

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15$$



$$= 8 + 16 \cdot 7 \quad = \quad 8 + 112 \quad = \quad 120.$$

We can extrapolate and come up with the formula for the line (n): $\dfrac{1}{2}\, n\,\big(n+1\big)$.

Q. How much is the following? (Justify your answer.)

$$\begin{array}{l} 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\ + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 \end{array}$$

— 210. Justification: It is $\dfrac{1}{2} \cdot 20 \cdot \big(20+1\big) = 210.$

Q. How much is the following? (Justify your answer.)

$$\begin{array}{l} 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\ + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 \\ + 21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 \\ + 31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39 + 40 \\ + 41 + 42 + 43 + 44 + 45 + 46 + 47 + 48 + 49 + 50 \\ + 51 + 52 + 53 + 54 + 55 + 56 + 57 + 58 + 59 + 60 \end{array}$$

— 1830. Justification: It is $\dfrac{1}{2} \cdot 60 \cdot \big(60+1\big) = 30 \cdot 61 = 1830.$

- §3. Double sum of consecutive integers.

Review. Analyze the patterns in the answers of the following:

(1) $1 \ = ?$
(2) $1 + 3 \ = ?$
(3) $1 + 3 + 6 \ = ?$
(4) $1 + 3 + 6 + 10 \ = ?$
(5) $1 + 3 + 6 + 10 + 15 \ = ?$
(6) $1 + 3 + 6 + 10 + 15 + 21 \ = ?$
(7) $1 + 3 + 6 + 10 + 15 + 21 + 28 \ = ?$
(8) $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 \ = ?$
(9) $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 \ = ?$
(10) $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 \ = ?$

— The heuristic way to figure out the answers for, say, (8):

(8) equals

$$\boxed{\begin{aligned} &\ \ 1 \\ &+1+2 \\ &+1+2+3 \\ &+1+2+3+4 \\ &+1+2+3+4+5 \\ &+1+2+3+4+5+6 \\ &+1+2+3+4+5+6+7 \\ &+1+2+3+4+5+6+7+8 \end{aligned}} \quad \longleftarrow \quad \underline{\underline{\text{Let's call it}}} \ \underline{\underline{\boxed{x}}}$$

though it is still helpful to remember

$$\boxed{x} = \boxed{1+3+6+10+15+21+28+36} \ .$$

Agree that x equals

$$\underbrace{\boxed{\begin{array}{l} 1 \quad + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \\ + 1 + 2 \quad + 3 + 4 + 5 + 6 + 7 + 8 + 9 \\ + 1 + 2 + 3 \quad + 4 + 5 + 6 + 7 + 8 + 9 \\ + 1 + 2 + 3 + 4 \quad + 5 + 6 + 7 + 8 + 9 \\ + 1 + 2 + 3 + 4 + 5 \quad + 6 + 7 + 8 + 9 \\ + 1 + 2 + 3 + 4 + 5 + 6 \quad + 7 + 8 + 9 \\ + 1 + 2 + 3 + 4 + 5 + 6 + 7 \quad + 8 + 9 \\ + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \quad + 9 \end{array}}}_{\text{Box \#1}} - \underbrace{\boxed{\begin{array}{r} 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \\ + 3 + 4 + 5 + 6 + 7 + 8 + 9 \\ + 4 + 5 + 6 + 7 + 8 + 9 \\ + 5 + 6 + 7 + 8 + 9 \\ + 6 + 7 + 8 + 9 \\ + 7 + 8 + 9 \\ + 8 + 9 \\ + 9 \end{array}}}_{\text{Box \#2}}$$

So we calculate Box #1 and Box #2 separately. (Sounds familiar?)

First, Box #1 is calculated simply as

$$\left.\begin{array}{l} \boxed{\ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\ \boxed{+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\ \boxed{+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\ \boxed{+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\ \boxed{+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\ \boxed{+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\ \boxed{+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\ \boxed{+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \end{array}\right] \begin{array}{c} \text{eight} \\ \text{identical} \\ \text{boxes,} \\ \text{each} \\ \text{equals} \\ 45 \end{array}$$

So

$$\underline{\text{Box \#1}} \ = \ 45 \cdot 8 \ = \ 360.$$

Meanwhile, Box #2 equals

$$
\begin{array}{cccccccc}
\boxed{2} & \boxed{\begin{array}{c} +\ 3 \\ +\ 3 \end{array}} & \boxed{\begin{array}{c} +\ 4 \\ +\ 4 \\ +\ 4 \end{array}} & \boxed{\begin{array}{c} +\ 5 \\ +\ 5 \\ +\ 5 \\ +\ 5 \end{array}} & \boxed{\begin{array}{c} +\ 6 \\ +\ 6 \\ +\ 6 \\ +\ 6 \\ +\ 6 \end{array}} & \boxed{\begin{array}{c} +\ 7 \\ +\ 7 \\ +\ 7 \\ +\ 7 \\ +\ 7 \\ +\ 7 \end{array}} & \boxed{\begin{array}{c} +\ 8 \\ +\ 8 \\ +\ 8 \\ +\ 8 \\ +\ 8 \\ +\ 8 \\ +\ 8 \end{array}} & \boxed{\begin{array}{c} +\ 9 \\ +\ 9 \\ +\ 9 \\ +\ 9 \\ +\ 9 \\ +\ 9 \\ +\ 9 \\ +\ 9 \end{array}} \\
\hline
2 & +\ 6 & +\ 12 & +\ 20 & +\ 30 & +\ 42 & +\ 56 & +\ 72 \\
\| & \| & \| & \| & \| & \| & \| & \| \\
2 \cdot \underline{\underline{1}} & 2 \cdot \underline{\underline{3}} & 2 \cdot \underline{\underline{6}} & 2 \cdot \underline{\underline{10}} & 2 \cdot \underline{\underline{15}} & 2 \cdot \underline{\underline{21}} & 2 \cdot \underline{\underline{28}} & 2 \cdot \underline{\underline{36}}
\end{array}
$$

+)

In other words, Box #2 equals

$$2 \quad \Big(\underbrace{1 + 3 + 6 + 10 + 15 + 21 + 28 + 36}_{\substack{\| \\ x}} \Big)$$

Thus

$$\underline{\text{Box \#2}} \quad = \quad 2\,x.$$

Remember,

$$x \quad = \quad \underset{\substack{\| \\ 360}}{\underline{\text{Box \#1}}} \quad - \quad \underset{\substack{\| \\ 2\,x}}{\underline{\text{Box \#2}}}$$

So this reads

$$x \quad = \quad 360 \; - \; 2\,x.$$

This is readily solved as

$$x \;=\; 120.$$

You can do the same for Line (n) instead of Line (8). You end up solving the equation

$$x \;=\; \frac{1}{2}\, n\Big(n+1\Big)\Big(n+2\Big) \;-\; 2\,x.$$

This is readily solved as $x \;=\; \frac{1}{6}\, n\Big(n+1\Big)\Big(n+2\Big)$. This way you come up with the formula for the line (n):

Formula. Let n be a positive integer. Then

$$\boxed{1 + \underbrace{3}_{\substack{\| \\ 1+2}} + \underbrace{6}_{\substack{\| \\ 1+2+3}} + \underbrace{10}_{\substack{\| \\ 1+2+3+4}} + \cdots + \underbrace{\frac{1}{2} n\Big(n+1\Big)}_{\substack{\| \\ 1+2+3+4+\cdots+n}} \;=\; \frac{1}{6}\, n\Big(n+1\Big)\Big(n+2\Big).}$$

Formula paraphrased. Let n be a positive integer. Then

$$\boxed{\begin{array}{l} \;\;1 \\ +\,1+2 \\ +\,1+2+3 \\ +\,1+2+3+4 \\ +\,1+2+3+4+5 \\ \;\;\vdots \qquad\qquad\qquad\qquad \ddots \\ +\,1+2+3+4+5+\cdots+\Big(n-1\Big) \\ +\,1+2+3+4+5+\cdots+\Big(n-1\Big)+n \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad = \frac{1}{6}\, n\Big(n+1\Big)\Big(n+2\Big). \end{array}}$$

Q. Find Line (70) in the above list. Justify your answer.

$$
\begin{aligned}
(70)\quad & 1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 \\
& + 66 + 78 + 91 + 105 + 120 + 136 + 153 + 171 + 190 + 210 \\
& + 231 + 253 + 276 + 300 + 325 + 351 + 378 + 406 + 435 + 465 \\
& + 496 + 528 + 561 + 595 + 630 + 666 + 703 + 741 + 780 + 820 \\
& + 861 + 903 + 946 + 990 + 1035 + 1081 + 1128 + 1176 + 1225 + 1275 \\
& + 1326 + 1378 + 1431 + 1485 + 1540 + 1596 + 1653 + 1711 + 1770 + 1830 \\
& + 1891 + 1953 + 2016 + 2080 + 2145 + 2211 + 2278 + 2346 + 2415 + 2485
\end{aligned}
$$

– 59640. Justification: It is

$$\frac{1}{6} \cdot 70 \cdot \Big(70+1\Big) \cdot \Big(70+2\Big) = \frac{1}{6} \cdot 70 \cdot 71 \cdot 72 \cdot = 59640.$$

Or, the same to say:

$$
\begin{aligned}
& 1 \\
& + 1 + 2 \\
& + 1 + 2 + 3 \\
& + 1 + 2 + 3 + 4 \\
& + 1 + 2 + 3 + 4 + 5 \\
& \vdots \qquad\qquad\qquad \ddots \\
& + 1 + 2 + 3 + 4 + 5 + \cdots + 69 \\
& + 1 + 2 + 3 + 4 + 5 + \cdots + 69 + 70 \qquad = \qquad 59640\ .
\end{aligned}
$$

- §4. Pascal's Triangle.

Review. Analyze the patterns in the answers of the following:

(1) $1 \;=?$
(2) $1 + 4 \;=?$
(3) $1 + 4 + 10 \;=?$
(4) $1 + 4 + 10 + 20 \;=?$
(5) $1 + 4 + 10 + 20 + 35 \;=?$
(6) $1 + 4 + 10 + 20 + 35 + 56 \;=?$
(7) $1 + 4 + 10 + 20 + 35 + 56 + 84 \;=?$
(8) $1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 \;=?$
(9) $1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 \;=?$

$\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \ddots$

(n) $1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + \cdots + \frac{1}{6}n\Big(n+1\Big)\Big(n+2\Big) \;=?$

— The heuristic way to figure out the answers for Line (n): Line (n) equals

$$\boxed{\begin{array}{l} \quad 1 \\ +1+3 \\ +1+3+6 \\ +1+3+6+10 \\ +1+3+6+10+15 \\ \quad\vdots \qquad\qquad\qquad \ddots \\ +1+3+6+10+15+\cdots+\frac{1}{2}n(n+1) \end{array}} \quad \longleftarrow \quad \underline{\underline{\text{Let's call it}}} \quad \underline{\underline{\boxed{x}}}$$

though it is still helpful to remember

$$\boxed{x} = \boxed{1 + 4 + 10 + 20 + 35 + \cdots + \frac{1}{6}\,n\,\Big(n+1\Big)\,\Big(n+2\Big)} \,.$$

x equals

$$
= \boxed{\begin{array}{l}
1 \quad\quad + 3 + 6 + 10 + 15 + \quad \cdots \quad + \frac{1}{2}(n+1)(n+2) \\
+ 1 + 3 \quad\quad + 6 + 10 + 15 + \quad \cdots \quad + \frac{1}{2}(n+1)(n+2) \\
+ 1 + 3 + 6 \quad\quad + 10 + 15 + \quad \cdots \quad + \frac{1}{2}(n+1)(n+2) \\
+ 1 + 3 + 6 + 10 \quad\quad + 15 + \quad \cdots \quad + \frac{1}{2}(n+1)(n+2) \\
+ 1 + 3 + 6 + 10 + 15 \quad\quad + \quad \cdots \quad + \frac{1}{2}(n+1)(n+2) \\
\quad \vdots \qquad\qquad\qquad\qquad \ddots \qquad\qquad \vdots \\
+ 1 + 3 + 6 + 10 + \cdots + \frac{1}{2}n(n+1) \quad\quad + \frac{1}{2}(n+1)(n+2)
\end{array}} \qquad \underline{\text{Box } \#1}
$$

$$
- \boxed{\begin{array}{r}
3 + 6 + 10 + 15 + \cdots + \frac{1}{2}(n+1)(n+2) \\
+ 6 + 10 + 15 + \cdots + \frac{1}{2}(n+1)(n+2) \\
+ 10 + 15 + \cdots + \frac{1}{2}(n+1)(n+2) \\
+ 15 + \cdots + \frac{1}{2}(n+1)(n+2) \\
\ddots \qquad\quad \vdots \quad \\
+ \frac{1}{2}(n+1)(n+2)
\end{array}} \qquad \underline{\text{Box } \#2}
$$

First, Box #1 is calculated simply as

$$
\left.\begin{array}{l}
\boxed{\quad 1 + 3 + 6 + 10 + 15 + \cdots + \frac{1}{2}(n+1)(n+2)} \\
\boxed{+ 1 + 3 + 6 + 10 + 15 + \cdots + \frac{1}{2}(n+1)(n+2)} \\
\boxed{+ 1 + 3 + 6 + 10 + 15 + \cdots + \frac{1}{2}(n+1)(n+2)} \\
\boxed{+ 1 + 3 + 6 + 10 + 15 + \cdots + \frac{1}{2}(n+1)(n+2)} \\
\qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
\boxed{+ 1 + 3 + 6 + 10 + 15 + \cdots + \frac{1}{2}(n+1)(n+2)}
\end{array}\right] \quad
\begin{array}{c}
n \\ \text{identical} \\ \text{boxes,} \\ \text{each} \\ \text{equals} \\ \frac{1}{6}(n+1)(n+2)(n+3) \\ \uparrow \\ \boxed{\begin{array}{c}\text{Shift } \; n \mapsto n+1 \; \text{ in} \\ \frac{1}{6}n(n+1)(n+2)\end{array}}
\end{array}
$$

So

$$
\begin{aligned}
\underline{\text{Box } \#1} \; &= \; n \cdot \frac{1}{6} \Big(n+1\Big) \Big(n+2\Big) \Big(n+3\Big) \\
&= \; \frac{1}{6}\, n \Big(n+1\Big) \Big(n+2\Big) \Big(n+3\Big) \, .
\end{aligned}
$$

Meanwhile, Box #2 equals

$$
\begin{array}{rcccccccl}
\boxed{3} & +\;6 & +\;10 & +\;15 & +\;21 & + & \cdots & + & \frac{1}{2}(n+1)(n+2) \\
 & +\;6 & +\;10 & +\;15 & +\;21 & + & \cdots & + & \frac{1}{2}(n+1)(n+2) \\
 & & +\;10 & +\;15 & +\;21 & + & \cdots & + & \frac{1}{2}(n+1)(n+2) \quad \longleftarrow \; \text{there are } n \text{ of } \frac{1}{2}(n+1)(n+2) \\
 & & & +\;15 & +\;21 & + & \cdots & + & \frac{1}{2}(n+1)(n+2) \\
 & & & & +\;21 & + & \cdots & + & \frac{1}{2}(n+1)(n+2) \\
 & & & & & \ddots & & & \quad \vdots \\
 & & & & & & & & \frac{1}{2}(n+1)(n+2) \\
\hline
+) \quad 3 & +\;12 & +\;30 & +\;60 & +\;105 & + & \cdots & + & \frac{1}{2}n(n+1)(n+2) \\
\| & \| & \| & \| & \| & & & & \| \\
3\cdot\underline{\underline{1}} & 3\cdot\underline{\underline{4}} & 3\cdot\underline{\underline{10}} & 3\cdot\underline{\underline{20}} & 3\cdot\underline{\underline{35}} & & & & 3\cdot\underline{\underline{\frac{1}{6}n(n+1)(n+2)}}
\end{array}
$$

In other words, Box #2 equals

$$
3 \quad \Bigg(\underbrace{1 \;+\; 4 \;+\; 10 \;+\; 20 \;+\; \cdots \;+\; \frac{1}{6}n(n+1)(n+2)}_{\substack{\| \\ x}} \Bigg)
$$

Thus

$$
\underline{\text{Box } \#2} \quad = \quad 3\,x.
$$

Remember,

$$x \;=\; \underbrace{\text{Box }\#1}_{\frac{1}{6}n(n+1)(n+2)(n+3)} \;-\; \underbrace{\text{Box }\#2}_{3\,x}$$

So this reads

$$x \quad = \quad \frac{1}{6}\, n\,\Big(n+1\Big)\,\Big(n+2\Big)\,\Big(n+3\Big) \quad - \quad 3\,x \qquad \Big(\underline{\text{key equation}}\Big)$$

This is readily solved as $x \;=\; \dfrac{1}{24}\, n\,\Big(n+1\Big)\,\Big(n+2\Big)\,\Big(n+3\Big).$ This way you come up with the formula:

Formula. Let n be a positive integer. Then

$$1 + \underbrace{4}_{1+3} + \underbrace{10}_{1+3+6} + \underbrace{20}_{1+3+6+10} + \cdots + \underbrace{\frac{1}{6}n\Big(n+1\Big)\Big(n+2\Big)}_{1+3+6+10+\cdots+\frac{1}{2}(n+1)(n+2)}$$

$$= \quad \frac{1}{24}\, n\,\Big(n+1\Big)\,\Big(n+2\Big)\,\Big(n+3\Big).$$

Formula paraphrased. Let n be a positive integer. Then

$$\begin{array}{lllllll}
1 & +1 & +1 & +1 & & +1 \\
 & +1+2 & +1+2 & +1+2 & & +1+2 \\
 & & +1+2+3 & +1+2+3 & \cdots & +1+2+3 \\
 & & & +1+2+3+4 & & +1+2+3+4 \\
 & & & & & \vdots \qquad \ddots \\
 & & & & & +1+2+3+4+\cdots+n
\end{array}$$

$$= \quad \frac{1}{24}\, n\,\Big(n+1\Big)\,\Big(n+2\Big)\,\Big(n+3\Big).$$

Review. If you continue the procedure, and consider

$$1 + 5 + 15 + 35 + 70 + 126 + 210 + 330 + \cdots + \frac{1}{24} n\Big(n+1\Big)\Big(n+2\Big)\Big(n+3\Big),$$

try to heuristically get hold of the formula for this, you will end up solving the new equation

$$x \quad = \quad \frac{1}{24}\, n\,\Big(n+1\Big)\,\Big(n+2\Big)\,\Big(n+3\Big)\,\Big(n+4\Big) \quad - \quad 4\,x.$$

This equation becomes

$$5\,x \quad = \quad \frac{1}{24}\, n\,\Big(n+1\Big)\,\Big(n+2\Big)\,\Big(n+3\Big)\,\Big(n+4\Big).$$

The answer is

$$\frac{1}{120} n\,\Big(n+1\Big)\,\Big(n+2\Big)\,\Big(n+3\Big)\,\Big(n+4\Big).$$

The fraction $\frac{1}{120}$ shows up as one fifth of $\frac{1}{24}$. Repeat: 120 arose as 5 times 24. But 24 arose as 4 times 6. Furthermore, 6 arose as 3 times 2. So, in short,

$$\begin{aligned} 120 &= 5 \cdot 24, \\ 24 &= 4 \cdot 6, \\ 6 &= 3 \cdot 2, \\ 2 &= 2 \cdot 1. \end{aligned}$$

From these

$$\begin{aligned} 120 &= 5 \cdot 24 \\ &= 5 \cdot 4 \cdot 6 \\ &= 5 \cdot 4 \cdot 3 \cdot 2 \\ &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! \quad \Big(\text{`five factorial'}\Big). \end{aligned}$$

By extrapolation, if you keep doing

$$
\begin{array}{rrrrrrrrrrr}
 & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & \cdots \\
\longrightarrow & 1, & 3, & 6, & 10, & 15, & 21, & 28, & 36, & 45, & \cdots \\
\longrightarrow & 1, & 4, & 10, & 20, & 35, & 56, & 84, & 120, & 165, & \cdots \\
\longrightarrow & 1, & 5, & 15, & 35, & 70, & 126, & 210, & 330, & 495, & \cdots \\
\longrightarrow & 1, & 6, & 21, & 56, & 126, & 252, & 462, & 792, & 1287, & \cdots \\
\longrightarrow & 1, & 7, & 28, & 84, & 210, & 462, & 924, & 1716, & 3003, & \cdots \\
\longrightarrow & \cdots
\end{array}
$$

then the formula for the numbers showing up in this list involves the factorial numbers in the denominators. More precisely, they are

$$
\begin{aligned}
\frac{n}{1} &= \frac{n}{1!}, \\
\frac{n\Big(n+1\Big)}{1\cdot 2} &= \frac{n\Big(n+1\Big)}{\cdot 2!}, \\
\frac{n\Big(n+1\Big)\Big(n+2\Big)}{1\cdot 2\cdot 3} &= \frac{n\Big(n+1\Big)\Big(n+2\Big)}{3!}, \\
\frac{n\Big(n+1\Big)\Big(n+2\Big)\Big(n+3\Big)}{1\cdot 2\cdot 3\cdot 4} &= \frac{n\Big(n+1\Big)\Big(n+2\Big)\Big(n+3\Big)}{4!}, \\
\frac{n\Big(n+1\Big)\Big(n+2\Big)\Big(n+3\Big)\Big(n+4\Big)}{1\cdot 2\cdot 3\cdot 4\cdot 5} &= \frac{n\Big(n+1\Big)\Big(n+2\Big)\Big(n+3\Big)\Big(n+4\Big)}{5!}, \\
\frac{n\Big(n+1\Big)\Big(n+2\Big)\Big(n+3\Big)\Big(n+4\Big)\Big(n+5\Big)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} & \\
&= \frac{n\Big(n+1\Big)\Big(n+2\Big)\Big(n+3\Big)\Big(n+4\Big)\Big(n+5\Big)}{6!},
\end{aligned}
$$

etc.

Below is called Pascal's triangle (which is a rearrangement of the above chart):

1 (row 0)

1 1 (row 1)

1 2 1 (row 2)

1 3 3 1 (row 3)

1 4 6 4 1 (row 4)

1 5 10 10 5 1 (row 5)

1 6 15 20 15 6 1 (row 6)

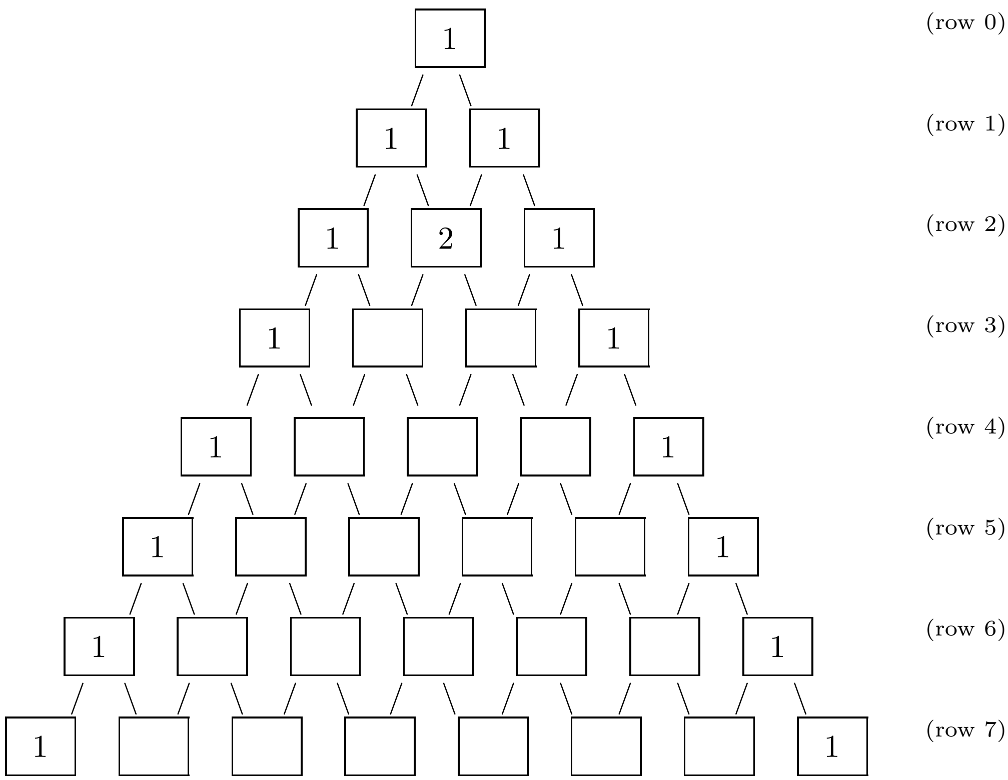
1 7 21 35 35 21 7 1 (row 7)

Rule.

At every spot, that number equals the sum of two numbers right above it.

Now, observe the symmetry. This pyramid possesses the left-and-right symmetry. And that's apparently thanks to the aforementioned rule. This explains why the same numbers show up twice at different spots (which was not easily seen from the previous chart). Now, the same rule actually allows us to algorithmically generate the next row, and so on and so forth. But then, suppose you need to know the numbers occupied in the 10-th row, 25-th row, 50-th row, 100-th row, and it takes a lot of time to manually do that. But guess what, you already know that there is a way to go around it. Rely on the formula. By the way, the above pyramid is so famous and so fundamental in math that it has a name. It is called the Pascal's triangle.

Q. Recover the emptied Pascal's triangle below:



Q. Find the third spot from the left in row 9 in the Pascal's triangle. (The left most spot in the same row is 1. That is is called the first spot from the left.)

— 36.

Q. Find the fourth spot from the left in row 10 in the Pascal's triangle. (The left most spot in the same row is 1. That is is called the first spot from the left.)

— 120.

Q. Write out the eighth spot from the left in row 20 in the Pascal's triangle, in a fraction form. You don't have to simplify the fraction. (The left most spot in the same row is 1. That is is called the first spot from the left.)

$$— \quad \frac{14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = \frac{14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20}{7!}.$$

Q. Write out the twelfth spot from the left in row 50 in the Pascal's triangle, in a fraction form. You don't have to simplify the fraction.

$$
\begin{aligned}
\text{—}\quad & \frac{40 \cdot 41 \cdot 42 \cdot 43 \cdot 44 \cdot 45 \cdot 46 \cdot 47 \cdot 48 \cdot 49 \cdot 50}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} \\
& = \frac{40 \cdot 41 \cdot 42 \cdot 43 \cdot 44 \cdot 45 \cdot 46 \cdot 47 \cdot 48 \cdot 49 \cdot 50}{11\,!}.
\end{aligned}
$$

- §5. Squares.

Review. $\boxed{x^2 = x \cdot x}$.

- Squares of integers between 0—19.

x	0	1	2	3	4	5	6	7	8	9
x^2	0	1	4	9	16	25	36	49	64	81

x	10	11	12	13	14	15	16	17	18	19
x^2	100	121	144	169	196	225	256	289	324	361

⋆ Important formations:

$$\left(x+1\right)^2, \qquad \left(x+2\right)^2, \qquad \left(x+3\right)^2, \qquad \left(x+4\right)^2, \qquad \textit{etc.}$$

Q. Substitute $\boxed{x = 7}$ in $\boxed{\left(x+4\right)^2}$. Calculate the result.

— $\left(7+4\right)^2 = 11^2 = 121.$

Q. Substitute $\boxed{x = 10}$ in $\boxed{\left(x+20\right)^2}$. Calculate the result.

— $\left(10+20\right)^2 = 30^2 = 900.$

- **Expansions.**

(a) $\left(x+\underline{\underline{1}}\right)^2 = x^2 + \underline{\underline{2}}\,x + \underline{\underline{1}}$.

(b) $\left(x+\underline{\underline{2}}\right)^2 = x^2 + \underline{\underline{4}}\,x + \underline{\underline{4}}$.

(c) $\left(x+\underline{\underline{3}}\right)^2 = x^2 + \underline{\underline{6}}\,x + \underline{\underline{9}}$.

(d) $\left(x+\underline{\underline{4}}\right)^2 = x^2 + \underline{\underline{8}}\,x + \underline{\underline{16}}$.

(e) $\left(x+\underline{\underline{5}}\right)^2 = x^2 + \underline{\underline{10}}\,x + \underline{\underline{25}}$.

(f) $\left(x+\underline{\underline{6}}\right)^2 = x^2 + \underline{\underline{12}}\,x + \underline{\underline{36}}$.

(g) $\left(x+\underline{\underline{7}}\right)^2 = x^2 + \underline{\underline{14}}\,x + \underline{\underline{49}}$.

(h) $\left(x+\underline{\underline{8}}\right)^2 = x^2 + \underline{\underline{16}}\,x + \underline{\underline{64}}$.

(i) $\left(x+\underline{\underline{9}}\right)^2 = x^2 + \underline{\underline{18}}\,x + \underline{\underline{81}}$.

(j) $\left(x+\underline{\underline{10}}\right)^2 = x^2 + \underline{\underline{20}}\,x + \underline{\underline{100}}$.

Meaning:

$\underline{\underline{\text{These identities are all true, no matter what number you substitute for } x.}}$

Patterns:

$$\left(x + \boxed{a}\right)^2 = x^2 + \boxed{2\,a}\,x + \boxed{a^2}$$

multiply by 2

square

⋆ Let's change the letter from a to y (for no compelling reason):

Formula A. $$\boxed{\left(x + y\right)^2 \;=\; x^2 \;+\; 2\,x\,y \;+\; y^2}\,.$$

Or use the letters a and b instead of x and y (for no compelling reason):

Formula A. $$\boxed{\left(a + b\right)^2 \;=\; a^2 \;+\; 2\,a\,b \;+\; b^2}\,.$$

Once again:

Meaning:

These formulas are all true, no matter what numbers you substitute for x, y, a and b each.

Q. Expand $\left(x+12\right)^2$.

— $x^2 + 24\,x + 144$.

Q. Expand $\left(x+15\right)^2$.

— $x^2 + 30\,x + 225$.

Q. Expand $\left(x+20\right)^2$.

— $x^2 + 40\,x + 400$.

$\star$ More important formulas:

Formula B.
$$\boxed{x^2 - y^2 = \left(x+y\right)\left(x-y\right)}\ .$$

$\star$ Here are some tweaks:

Formula C.
$$\boxed{\left(x+y\right)^2 - \left(x-y\right)^2 = 4\,x\,y}\ .$$

Formula D.
$$\boxed{\left(x^2+y^2\right)^2 - 4\,x^2\,y^2 = \left(x+y\right)^2\left(x-y\right)^2}\ .$$

Formula E.
$$\boxed{\begin{aligned}&\left(x^2+y^2+z^2\right)^2 - 4\left(x^2\,y^2 + x^2\,z^2 + y^2\,z^2\right)\\ &= \left(x+y+z\right)\cdot\left(x-y-z\right)\cdot\left(-x+y-z\right)\cdot\left(-x-y+z\right).\end{aligned}}$$

$\star$ Formula E has a connection with geometry. More precisely, the two sides of Formula E are precisely the negative of the square of the area of the triangle whose edges have lengths x, y and z (Heron's formula).

- §6. Cubes.

Review. $\boxed{x^3 \; = \; x \cdot x \cdot x}$.

Clearly

$$\boxed{x^3 \; = \; x^2 \cdot x}\ , \text{ or the same } \boxed{x^3 \; = \; x \cdot x^2}\ .$$

- Cubes of integers between 0—19.

x	0	1	2	3	4	5	6	7	8	9
x^3	0	1	8	27	64	125	216	343	512	729

x	10	11	12	13	14	15	16	17	18	19
x^3	1000	1331	1728	2197	2744	3375	4096	4913	5832	6859

⋆ Important formations:

$$\left(x+1\right)^3, \qquad \left(x+2\right)^3, \qquad \left(x+3\right)^3, \qquad \left(x+4\right)^3, \qquad \textit{etc.}$$

Q. Substitute $\boxed{x \; = \; 4}$ in $\boxed{\left(x+2\right)^3}$. Calculate the result.

— $\left(4+2\right)^3 \; = \; 6^3 \; = \; 216.$

- **Expansions.**

(a) $$\left(x+\underline{\underline{1}}\right)^3 = x^3 + \underline{\underline{3}}\,x^2 + \underline{\underline{3}}\,x + \underline{\underline{1}}\,.$$

(b) $$\left(x+\underline{\underline{2}}\right)^3 = x^3 + \underline{\underline{6}}\,x^2 + \underline{\underline{12}}\,x + \underline{\underline{8}}\,.$$

(c) $$\left(x+\underline{\underline{3}}\right)^3 = x^3 + \underline{\underline{9}}\,x^2 + \underline{\underline{27}}\,x + \underline{\underline{27}}\,.$$

(d) $$\left(x+\underline{\underline{4}}\right)^3 = x^3 + \underline{\underline{12}}\,x^2 + \underline{\underline{48}}\,x + \underline{\underline{64}}\,.$$

(e) $$\left(x+\underline{\underline{5}}\right)^3 = x^3 + \underline{\underline{15}}\,x^2 + \underline{\underline{75}}\,x + \underline{\underline{125}}\,.$$

(f) $$\left(x+\underline{\underline{6}}\right)^3 = x^3 + \underline{\underline{18}}\,x^2 + \underline{\underline{108}}\,x + \underline{\underline{216}}\,.$$

(g) $$\left(x+\underline{\underline{7}}\right)^3 = x^3 + \underline{\underline{21}}\,x^2 + \underline{\underline{147}}\,x + \underline{\underline{343}}\,.$$

(h) $$\left(x+\underline{\underline{8}}\right)^3 = x^3 + \underline{\underline{24}}\,x^2 + \underline{\underline{192}}\,x + \underline{\underline{512}}\,.$$

(i) $$\left(x+\underline{\underline{9}}\right)^3 = x^3 + \underline{\underline{27}}\,x^2 + \underline{\underline{243}}\,x + \underline{\underline{729}}\,.$$

(j) $$\left(x+\underline{\underline{10}}\right)^3 = x^3 + \underline{\underline{30}}\,x^2 + \underline{\underline{300}}\,x + \underline{\underline{1000}}\,.$$

Meaning:

$\underline{\underline{\text{These identities are all true, no matter what number you substitute for } x.}}$

Patterns: $$\left(x + \boxed{a} \right)^3 = x^3 + \boxed{3\,a}\, x^2 + \boxed{3\,a^2}\, x + \boxed{a^3}$$

multiply by 3

square, then multiply by 3

cube

$\star$ Let's change the letter from a to y (for no compelling reason):

Formula A.

$$\boxed{\left(x + y\right)^3 = x^3 + 3\,x^2\,y + 3\,x\,y^2 + y^3} \, .$$

Or use the letters a and b instead of x and y (for no compelling reason):

Formula A.

$$\boxed{\left(a + b\right)^3 = a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3} \, .$$

Once again:

Meaning:

These formulas are all true, no matter what numbers you substitute for x, y, a and b each.

Q. Expand $\left(x+2\right)^3$. (Don't take a peek at the previous page.)

— $x^3 + 6\,x^2 + 12\,x + 8$.

Q. Expand $\left(x+5\right)^3$. (Don't take a peek at the previous page.)

— $x^3 + 15\,x^2 + 75\,x + 125$.

More important formulas:

Formula B.

(a) $$\boxed{x^3 + y^3 = \left(x + y\right)\left(x^2 - x\,y + y^2\right)}\,,$$

(b) $$\boxed{x^3 - y^3 = \left(x - y\right)\left(x^2 + x\,y + y^2\right)}\,.$$

$\star$ Below are some tweaks (you don't have to memorize them):

Formula C.

$$\boxed{\begin{aligned} &\left(x + y + z\right)^3 \\ &\quad = x^3 + y^3 + z^3 + 3\left(x + y\right)\left(x + z\right)\left(y + z\right). \end{aligned}}$$

Formula D.

$$\boxed{\begin{aligned} &x^3 + y^3 + z^3 - 3\,x\,y\,z \\ &\quad = \left(x + y + z\right)\left(x^2 + y^2 + z^2 - x\,y - x\,z - y\,z\right). \end{aligned}}$$

Formula D′. If a, b and c satisfy

$$\boxed{a + b + c = 0},$$

then

$$\boxed{a^3 + b^3 + c^3 = 3\,a\,b\,c}.$$

Formula D″. If a, b, c, p, q and r satisfy

$$\boxed{\begin{aligned} p &= a + b, \\ q &= a + c, \qquad \text{and} \\ r &= b + c \end{aligned}}$$

then

$$\boxed{p^3 + q^3 + r^3 - 3\,p\,q\,r = 2\left(a^3 + b^3 + c^3 - 3\,a\,b\,c\right)}.$$

Q. Agree that if you substitute

$$a = 2, \qquad b = 3 \qquad \text{and} \qquad c = -5$$

in $a + b + c$, then the result is 0.

Then substitute

$$a = 2, \qquad b = 3 \qquad \text{and} \qquad c = -5$$

in each of $a^3 + b^3 + c^3$, and $3\,a\,b\,c$.

— $2 + 3 + \big(-5\big)$ clearly equals 0.

$$a^3 + b^3 + c^3 \quad = \quad 2^3 + 3^3 + \big(-5\big)^3 \quad = \quad 8 + 27 + \big(-125\big) \quad = \quad -90,$$

whereas $\qquad 3\,a\,b\,c \quad = \quad 3 \cdot 2 \cdot 3 \cdot \big(-5\big) \quad = \quad -90.$

Q. Substitute

$$a = 1, \qquad b = -2 \qquad \text{and} \qquad c = 1$$

in

$$\boxed{\begin{aligned} p &= a + b, \\ q &= a + c, \qquad\qquad \text{and} \\ r &= b + c \end{aligned}}$$

— $p = 1 + \big(-2\big) = -1, \qquad q = 1 + 1 = 2, \qquad r = \big(-2\big) + 1 = -1.$

Q. Substitute the values $a,\ b,\ c,\ p,\ q$ and r as in the previous question in each of

$$\boxed{p^3 + q^3 + r^3 - 3\,p\,q\,r} \qquad \text{and} \qquad \boxed{2\left(a^3 + b^3 + c^3 - 3\,a\,b\,c\right)}.$$

— $$p^3 + q^3 + r^3 - 3pqr = \left(-1\right)^3 + 2^3 + \left(-1\right)^3 - 3\cdot\left(-1\right)\cdot 2\cdot\left(-1\right)$$

$$= \left(-1\right) + 8 + \left(-1\right) - 6 = 0.$$

$$2\left(a^3 + b^3 + c^3 - 3abc\right) = 2\left(1 + \left(-2\right)^3 + 1^3 - 3\cdot 1\cdot\left(-2\right)\cdot 1\right)$$

$$= 2\left(1 + \left(-8\right) + 1 + 6\right) = 0.$$

• §7. Binomial formula.

Review. Generalizing

$$\boxed{\left(x + y\right)^2 = x^2 + 2\,x\,y + y^2},$$

and

$$\boxed{\left(x + y\right)^3 = x^3 + 3\,x^2\,y + 3\,x\,y^2 + y^3}.$$

to 'higher powers' requires to know Pascal triangle.

Definition. For a positive integer n, define x^n as

$$\boxed{x^n = \underbrace{x \cdot x \cdot x \cdot \quad \cdots \quad \cdot x}_{n}.}$$

For example:

$$\begin{aligned} x^{24} = \quad & x \cdot x \cdot x \cdot x \cdot x \cdot x \\ & \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \\ & \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \\ & \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x\,. \end{aligned}$$

⋆ For $n = 0$ and $n = 1$, we set

$$\boxed{x^0 = 1} \qquad \text{and}$$

$$\boxed{x^1 = x},$$

by convention.

10-to-the powers:

$$\boxed{10^n \;=\; 1\,\underbrace{0\,0\,0\,0\,0\;\ldots\;0}_{n}}\ .$$

$\star$ In mathematics, we don't place ' , ' (comma) after every third digit.

- **0-to-the-powers.**

$$\begin{aligned} 0^1 &= 0, \\ 0^2 &= 0, \\ 0^3 &= 0, \\ 0^4 &= 0, \\ 0^5 &= 0, \\ 0^6 &= 0, \\ 0^7 &= 0, \\ \vdots & \quad \vdots \end{aligned}$$

- **1-to-the-powers.**

$$\begin{aligned} 1^1 &= 1, \\ 1^2 &= 1, \\ 1^3 &= 1, \\ 1^4 &= 1, \\ 1^5 &= 1, \\ 1^6 &= 1, \\ 1^7 &= 1, \\ \vdots & \quad \vdots \end{aligned}$$

$$\boxed{1^0 = 1} \quad \text{whereas} \quad \underline{\underline{\boxed{0^0}}} \quad \underline{\underline{\text{is undefined}}}\ .$$

- **$\left(-1\right)$-to-the-powers.**

$$
\begin{aligned}
\left(-1\right)^1 &= -1,\\
\left(-1\right)^2 &= 1,\\
\left(-1\right)^3 &= -1,\\
\left(-1\right)^4 &= 1,\\
\left(-1\right)^5 &= -1,\\
\left(-1\right)^6 &= 1,\\
\left(-1\right)^7 &= -1,\\
\left(-1\right)^8 &= 1,\\
\left(-1\right)^9 &= -1,\\
\left(-1\right)^{10} &= 1,\\
\vdots \qquad & \quad \vdots
\end{aligned}
$$

In short,

$$\boxed{\left(-1\right)^n = \begin{cases} 1 & \left(\text{if } n \text{ is } \underline{\underline{\text{even}}}\right), \\ -1 & \left(\text{if } n \text{ is } \underline{\underline{\text{odd}}}\right). \end{cases}}$$

- $\boldsymbol{\left(-a\right)}$**-to-the-powers.**

$$
\begin{aligned}
\left(-a\right)^1 &= -a, \\
\left(-a\right)^2 &= a^2, \\
\left(-a\right)^3 &= -a^3, \\
\left(-a\right)^4 &= a^4, \\
\left(-a\right)^5 &= -a^5, \\
\left(-a\right)^6 &= a^6, \\
\left(-a\right)^7 &= -a^7, \\
\left(-a\right)^8 &= a^8, \\
\left(-a\right)^9 &= -a^9, \\
\left(-a\right)^{10} &= a^{10}, \\
&\vdots
\end{aligned}
$$

In short,

$$
\boxed{\left(-a\right)^n = \begin{cases} a^n & \left(\text{if } n \text{ is } \underline{\underline{\text{even}}}\right), \\ -a^n & \left(\text{if } n \text{ is } \underline{\underline{\text{odd}}}\right). \end{cases}}
$$

Q. $2^4 =?$

— 16.

Q. $3^5 =?$

— 243.

Q. $10^2 =?$

— 100.

Q. $1^7 =?$

— 1.

Q. $\left(-1\right)^{10} =?$

— 1.

Q. $\left(-1\right)^{25} =?$

— -1.

Q. $\left(-3\right)^{3} =?$

— -27.

Q. $\left(-2\right)^{6} =?$

— 64.

- **2-to-the-powers.**

  The numbers in the following sequence are called " 2-to-the-powers ":

$$\begin{aligned}
2^1 &= 2,\\
2^2 &= 4,\\
2^3 &= 8,\\
2^4 &= 16,\\
2^5 &= 32,\\
2^6 &= 64,\\
2^7 &= 128,\\
2^8 &= 256,\\
2^9 &= 512,\\
2^{10} &= 1024,\\
2^{11} &= 2048,\\
2^{12} &= 4096,\\
2^{13} &= 8192,\\
2^{14} &= 16384,\\
2^{15} &= 32768,\\
2^{16} &= 65536,\\
&\vdots
\end{aligned}$$

2-to-the-powers frequently appear in mathematics. Please familiarize yourself with the above listed numbers (the first sixteen of 2-to-the-powers).

Q. Identify all 2-to-the-powers among the numbers listed below. Write each of those 2-to-the-powers in the form 2^n with a concrete positive integer n.

$$\begin{array}{rrrrrrrrr}
8, & 12, & 24, & 32, & 48, & 64, & 80, & 84, & 128,\\
144, & 216, & 256, & 360, & 384, & 480, & 512, & 768, & 784,\\
1296, & 1440, & 2016, & 2048, & 2560, & 3840, & 5040, & 6912, & 8192.
\end{array}$$

— 8, 32, 64, 128, 256, 512, 2048, 8192.

$$8 = 2^3, \qquad 32 = 2^5, \qquad 64 = 2^6, \qquad 128 = 2^7,$$
$$256 = 2^8, \qquad 512 = 2^9, \qquad 2048 = 2^{11}, \qquad 8192 = 2^{13}.$$

- **Binomial Formula.**

Finally today's main theme. Check this out:

$$
\begin{aligned}
&(1) \quad \left(x + y\right)^1 = x + y,\\
&(2) \quad \left(x + y\right)^2 = x^2 + \underline{\underline{2}}\, x\, y + y^2,\\
&(3) \quad \left(x + y\right)^3 = x^3 + \underline{\underline{3}}\, x^2 y + \underline{\underline{3}}\, x\, y^2 + y^3,\\
&(4) \quad \left(x + y\right)^4 = x^4 + \underline{\underline{4}}\, x^3 y + \underline{\underline{6}}\, x^2 y^2 + \underline{\underline{4}}\, x\, y^3 + y^4,\\
&(5) \quad \left(x + y\right)^5 = x^5 + \underline{\underline{5}}\, x^4 y + \underline{\underline{10}}\, x^3 y^2 + \underline{\underline{10}}\, x^2 y^3 + \underline{\underline{5}}\, x\, y^4 + y^5,\\
&(6) \quad \left(x + y\right)^6 = x^6 + \underline{\underline{6}}\, x^5 y + \underline{\underline{15}}\, x^4 y^2 + \underline{\underline{20}}\, x^3 y^3 + \underline{\underline{15}}\, x^2 y^4 + \underline{\underline{6}}\, x\, y^5 + y^6,\\
&\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddots
\end{aligned}
$$

Get used to the term 'coefficient'.

- The coefficient of $x^3 y$ in the right-hand side of (4) is 4.
- The coefficient of $x^2 y^3$ in the right-hand side of (5) is 10.
- The coefficient of y^6 in the right-hand side of (6) is 1.

The coefficients in the above expansion formulas are nothing but the numbers showing up in the Pascal. Those numbers are called binomial coefficients .

Indeed, they are

(1) 1, 1.

(2) 1, 2, 1.

(3) 1, 3, 3, 1.

(4) 1, 4, 6, 4, 1.

(5) 1, 5, 10, 10, 5, 1.

(6) 1, 6, 15, 20, 15, 6, 1.

The above coincides with row 0 –row 6 of Pascal.

Q. Expand $\left(x + y\right)^7$.

— The (row 7) in the Pascal is

(7) 1, 7, 21, 35, 35, 21, 7, 1.

So

(7) $\left(x + y\right)^7$

$$= x^7 + \underline{\underline{7}}\,x^6 y + \underline{\underline{21}}\,x^5 y^2 + \underline{\underline{35}}\,x^4 y^3 + \underline{\underline{35}}\,x^3 y^4 + \underline{\underline{21}}\,x^2 y^5 + \underline{\underline{7}}\,x\,y^6 + y^7.$$

Q. Expand $\left(x + y\right)^8$.

— The (row 8) in the Pascal is

(8) 1, 8, 28, 56, 70, 56, 28, 8, 1.

So

(8) $\left(x + y\right)^8$

$$= x^8 + \underline{\underline{8}}\,x^7 y + \underline{\underline{28}}\,x^6 y^2 + \underline{\underline{56}}\,x^5 y^3 + \underline{\underline{70}}\,x^4 y^4 + \underline{\underline{56}}\,x^3 y^5 + \underline{\underline{35}}\,x^2 y^6 + \underline{\underline{8}}\,x\,y^7 + y^8.$$

Notation (binomial coefficient). In the Pascal's triangle:

- The numbers in row 1, from left to right, are denoted as

$$\begin{matrix} \binom{1}{0}, & \binom{1}{1}. \\ \| & \| \\ 1 & 1 \end{matrix}$$

- The numbers in row 2, from left to right, are denoted as

$$\begin{matrix} \binom{2}{0}, & \binom{2}{1}, & \binom{2}{2}. \\ \| & \| & \| \\ 1 & 2 & 1 \end{matrix}$$

- The numbers in row 3, from left to right, are denoted as

$$\begin{matrix} \binom{3}{0}, & \binom{3}{1}, & \binom{3}{2}, & \binom{3}{3}. \\ \| & \| & \| & \| \\ 1 & 3 & 3 & 1 \end{matrix}$$

- The numbers in row 4, from left to right, are denoted as

$$\begin{matrix} \binom{4}{0}, & \binom{4}{1}, & \binom{4}{2}, & \binom{4}{3}, & \binom{4}{4}. \\ \| & \| & \| & \| & \| \\ 1 & 4 & 6 & 4 & 1 \end{matrix}$$

- The numbers in row 5, from left to right, are denoted as

$$\begin{matrix} \binom{5}{0}, & \binom{5}{1}, & \binom{5}{2}, & \binom{5}{3}, & \binom{5}{4}, & \binom{5}{5}. \\ \| & \| & \| & \| & \| & \| \\ 1 & 5 & 10 & 10 & 5 & 1 \end{matrix}$$

$$\begin{matrix} \vdots & & & & & & \ddots \end{matrix}$$

More generally, in the Pascal:

◦ The numbers in row n, from left to right, are denoted as

$$\binom{n}{0}, \qquad \binom{n}{1}, \qquad \binom{n}{2}, \qquad \binom{n}{3}, \qquad \cdots \qquad \binom{n}{n}.$$

These are called $\underline{\underline{\text{ the binomial coefficients }}}$. Note

$$\boxed{\binom{n}{0} = 1,} \qquad \boxed{\binom{n}{n} = 1.}$$

Q. $\binom{5}{0} =?$ $\quad \binom{6}{2} =?$ $\quad \binom{6}{3} =?$

$\binom{7}{2} =?$ $\quad \binom{7}{4} =?$ $\quad \binom{8}{4} =?$

— $\binom{5}{0} = 1.$ $\quad \binom{6}{2} = 15.$ $\quad \binom{6}{3} = 20.$

$\binom{7}{2} = 21.$ $\quad \binom{7}{4} = 35.$ $\quad \binom{8}{4} = 70.$

• Using this new notation, we can rewrite the previous formulas as

$$\begin{aligned}
&(1) \quad \left(x+y\right)^1 = \binom{1}{0} x + \binom{1}{1} y, \\
&(2) \quad \left(x+y\right)^2 = \binom{2}{0} x^2 + \binom{2}{1} x\,y + \binom{2}{2} y^2, \\
&(3) \quad \left(x+y\right)^3 = \binom{3}{0} x^3 + \binom{3}{1} x^2\,y + \binom{3}{2} x\,y^2 + \binom{3}{3} y^3, \\
&(4) \quad \left(x+y\right)^4 = \binom{4}{0} x^4 + \binom{4}{1} x^3\,y + \binom{4}{2} x^2\,y^2 + \binom{4}{3} x\,y^3 + \binom{4}{4} y^4, \\
&\qquad\quad \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddots
\end{aligned}$$

Formula A (binomial coefficients).

Let n and k be integers, with $0 < k < n$. Then

$$\boxed{\binom{n}{k} = \frac{n\ \big(n-1\big)\ \big(n-2\big)\ \cdots\ \big(n-k+1\big)}{1\ \cdot\ 2\ \cdot\ 3\ \cdot\ \cdots\ \cdot\ k}}\ .$$

Or, alternatively:

$$\boxed{\binom{n}{k} = \frac{n\ \big(n-1\big)\ \big(n-2\big)\ \cdots\ \big(n-k+1\big)}{k\,!}}\ .$$

Q. Write $\binom{9}{5}$ in the fraction form. You don't have to calculate the answers.

— $\binom{9}{5} = \dfrac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \dfrac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{5\,!}$.

Q. Write $\binom{10}{3}$ in the fraction form. You don't have to calculate the answers.

— $\binom{10}{3} = \dfrac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = \dfrac{10 \cdot 9 \cdot 8}{3\,!}$.

Q. Write $\binom{18}{6}$ in the fraction form. You don't have to calculate the answers.

— $\binom{18}{6} = \dfrac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \dfrac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}{6\,!}$.

Formula B (Binomial Formula). Let n be a positive integer. Then

$$\begin{aligned}
&\big(x+y\big)^n \\
&= \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \binom{n}{3} x^{n-3}y^3 + \cdots \\
&\quad + \binom{n}{n-3} x^3\, y^{n-3} + \binom{n}{n-2} x^2\, y^{n-2} + \binom{n}{n-1} x\, y^{n-1} + \binom{n}{n} y^n.
\end{aligned}$$

Q. Spell out the binomial formula for each of

$$\big(x+y\big)^5, \qquad \big(x+y\big)^8, \qquad \text{and} \qquad \big(x+y\big)^9.$$

First give the formula that includes the notation $\binom{n}{k}$. Then convert those $\binom{n}{k}$ into numbers and rewrite your answer accordingly.

— $\big(x+y\big)^5$

$$\begin{aligned}
&= \binom{5}{0} x^5 + \binom{5}{1} x^4\, y + \binom{5}{2} x^3\, y^2 + \binom{5}{3} x^2\, y^3 + \binom{5}{4} x\, y^4 + \binom{5}{5} y^5 \\
&= x^5 + 5\ x^4\, y + 10\ x^3\, y^2 + 10\ x^2\, y^3 + 5\ x\, y^4 + y^5,
\end{aligned}$$

$$\begin{aligned}
&\big(x+y\big)^8 \\
&= \binom{8}{0} x^8 + \binom{8}{1} x^7\, y + \binom{8}{2} x^6\, y^2 + \binom{8}{3} x^5\, y^3 + \binom{8}{4} x^4\, y^4 \\
&\quad + \binom{8}{5} x^3\, y^5 + \binom{8}{6} x^2\, y^6 + \binom{8}{7} x\, y^7 + \binom{8}{8} y^8 \\
&= x^8 + 8\ x^7\, y + 28\, x^6\, y^2 + 56\ x^5\, y^3 + 70\, x^4\, y^4 \\
&\quad + 56\, x^3\, y^5 + 28\ x^2\, y^6 + 8\ x\, y^7 + y^8.
\end{aligned}$$

$$\left(x + y\right)^9$$

$$= \binom{9}{0} x^9 + \binom{9}{1} x^8 y + \binom{9}{2} x^7 y^2 + \binom{9}{3} x^6 y^3 + \binom{9}{4} x^5 y^4$$

$$+ \binom{9}{5} x^4 y^5 + \binom{9}{6} x^3 y^6 + \binom{9}{7} x^2 y^7 + \binom{9}{8} x y^8 + \binom{9}{9} y^9$$

$$= x^9 + 9 x^8 y + 36 x^7 y^2 + 84 x^6 y^3 + 126 x^5 y^4$$

$$+ 126 x^4 y^5 + 84 x^3 y^6 + 36 x^2 y^7 + 9 x y^8 + y^9.$$

Q. How much does it make it you add up the numbers in one whole row in Pascal?

— Heuristically:

Row 1: $1 + 1 = 2.$

Row 2: $1 + 2 + 1 = 4.$

Row 3: $1 + 3 + 3 + 1 = 8.$

Row 4: $1 + 4 + 6 + 4 + 1 = 16.$

Row 5: $1 + 5 + 10 + 10 + 5 + 1 = 32.$

Row 6: $1 + 6 + 15 + 20 + 15 + 6 + 1 = 64.$

Row 7: $1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128.$

These are 2-to-the-powers. So, the projected result is as follows: the sum of all nubmers in (row n) of the Pascal equals 2^n. Now, proof:

Substituting $\boxed{x = 1}$ and $\boxed{y = 1}$ in the Binomial Formula yields

$$\begin{aligned}
\left(1+1\right)^n &= \binom{n}{0} \cdot 1^n + \binom{n}{1} \cdot 1^{n-1} \cdot 1^1 + \binom{n}{2} \cdot 1^{n-2} \cdot 1^2 + \cdots \\
&\quad + \binom{n}{n-2} \cdot 1^2 \cdot 1^{n-2} + \binom{n}{n-1} \cdot 1^1 \cdot 1^{n-1} + \binom{n}{n} \cdot 1^n \\
&= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n}.
\end{aligned}$$

• §8. More about Binomial Coefficients.

Review. The Pascal's triangle, using the notation just introduced is

$$\begin{array}{c}
\boxed{\binom{0}{0}} \\
/ \quad \backslash \\
\boxed{\binom{1}{0}} \quad \boxed{\binom{1}{1}} \\
/ \quad \backslash \quad / \quad \backslash \\
\boxed{\binom{2}{0}} \quad \boxed{\binom{2}{1}} \quad \boxed{\binom{2}{2}} \\
/ \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \\
\boxed{\binom{3}{0}} \quad \boxed{\binom{3}{1}} \quad \boxed{\binom{3}{2}} \quad \boxed{\binom{3}{3}} \\
/ \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \\
\boxed{\binom{4}{0}} \quad \boxed{\binom{4}{1}} \quad \boxed{\binom{4}{2}} \quad \boxed{\binom{4}{3}} \quad \boxed{\binom{4}{4}} \\
/ \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \\
\boxed{\binom{5}{0}} \quad \boxed{\binom{5}{1}} \quad \boxed{\binom{5}{2}} \quad \boxed{\binom{5}{3}} \quad \boxed{\binom{5}{4}} \quad \boxed{\binom{5}{5}} \\
/ \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \\
\boxed{\binom{6}{0}} \quad \boxed{\binom{6}{1}} \quad \boxed{\binom{6}{2}} \quad \boxed{\binom{6}{3}} \quad \boxed{\binom{6}{4}} \quad \boxed{\binom{6}{5}} \quad \boxed{\binom{6}{6}} \\
/ \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash \quad / \quad \backslash
\end{array}$$

What are in the boxes are actual numbers, and they are dictated by

Formula A (binomial coefficients).

Let n and k be integers, with $0 < k < n$. Then

$$\boxed{\binom{n}{k} = \frac{n\,\big(n-1\big)\,\big(n-2\big)\,\cdots\,\big(n-k+1\big)}{1\cdot 2\cdot 3\cdot\cdots\cdot k}}\,.$$

Or

$$\boxed{\binom{n}{k} = \frac{n\,\big(n-1\big)\,\big(n-2\big)\,\cdots\,\big(n-k+1\big)}{k\,!}}\,.$$

Using this formula we recover the Pascal:

							1									(row 0)
						1		1								(row 1)
					1		2		1							(row 2)
				1		3		3		1						(row 3)
			1		4		6		4		1					(row 4)
		1		5		10		10		5		1				(row 5)
	1		6		15		20		15		6		1			(row 6)
1		7		21		35		35		21		7		1		(row 7)

But the real upshot of Formula A above is it tells you the actual number at any spot of Pascal, no matter how far down it is.

⋆ Properties and rules of the binomial corfficients.

$$\binom{0}{0} = 1,$$

$$\binom{1}{0} = 1, \qquad \binom{1}{1} = 1,$$

$$\binom{2}{0} = 1, \qquad \binom{2}{2} = 1,$$

$$\binom{3}{0} = 1, \qquad \binom{3}{3} = 1,$$

$$\binom{4}{0} = 1, \qquad \binom{4}{4} = 1,$$

$$\binom{5}{0} = 1, \qquad \binom{5}{5} = 1,$$

$$\binom{6}{0} = 1, \qquad \binom{6}{6} = 1,$$

$$\binom{7}{0} = 1, \qquad \binom{7}{7} = 1,$$

$$\vdots \qquad \vdots$$

A way to write these simultaneously in a short form is as follows:

Initial Conditions (of the Pascal algorithm).

$$\boxed{\binom{n}{0} = 1}, \qquad \boxed{\binom{n}{n} = 1}$$

$$\Big(n = 0, 1, 2, 3, \cdots\Big).$$

$\star$ Now, let's recite the rule of the Pascal:

Rule.

$$\underline{\underline{\text{At every spot, that number equals the sum of two numbers right above it.}}}$$

This is translated into

$$\binom{2}{1} = \binom{1}{0} + \binom{1}{1},$$

$$\binom{3}{1} = \binom{2}{0} + \binom{2}{1}, \quad \binom{3}{2} = \binom{2}{1} + \binom{2}{2},$$

$$\binom{4}{1} = \binom{3}{0} + \binom{3}{1}, \quad \binom{4}{2} = \binom{3}{1} + \binom{3}{2}, \quad \binom{4}{3} = \binom{3}{2} + \binom{3}{3},$$

$$\binom{5}{1} = \binom{4}{0} + \binom{4}{1}, \quad \binom{5}{2} = \binom{4}{1} + \binom{4}{2}, \quad \binom{5}{3} = \binom{4}{2} + \binom{4}{3}, \quad \binom{5}{4} = \binom{4}{3} + \binom{4}{4},$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \ddots$$

A way to write these simultaneously in a short form is

$$\boxed{\binom{n+1}{1} = \binom{n}{0} + \binom{n}{1}}, \quad \boxed{\binom{n+1}{2} = \binom{n}{1} + \binom{n}{2}}, \quad \boxed{\binom{n+1}{3} = \binom{n}{2} + \binom{n}{3}}, \quad \cdots$$

$$\Big(n = 0, 1, 2, 3, \cdots\Big).$$

Or, just simply

Rule *k*. $$\boxed{\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}} \qquad \Big(0 < k \leq n\Big).$$

The symmetry the Pascal exhibits is translated into

$$
\begin{array}{lll}
\binom{3}{1} = \binom{3}{2}, & & \\
\binom{4}{1} = \binom{4}{3}, & & \\
\binom{5}{1} = \binom{5}{4}, & \binom{5}{2} = \binom{5}{3}, & \\
\binom{6}{1} = \binom{6}{5}, & \binom{6}{2} = \binom{6}{4}, & \\
\binom{7}{1} = \binom{7}{6}, & \binom{7}{2} = \binom{7}{5}, & \binom{7}{3} = \binom{7}{4}, \\
\binom{8}{1} = \binom{8}{7}, & \binom{8}{2} = \binom{8}{6}, & \binom{8}{3} = \binom{8}{5}, \\
\vdots & \vdots & \vdots
\end{array}
$$

A way to write these simultaneously in a short form is

$$
\boxed{\binom{n}{1} = \binom{n}{n-1}}, \quad \boxed{\binom{n}{2} = \binom{n}{n-2}}, \quad \boxed{\binom{n}{3} = \binom{n}{n-3}}, \quad \cdots
$$

$$
\Big(n = 0, 1, 2, 3, \cdots\Big).
$$

Or, just simply

Symmetry. $$\boxed{\binom{n}{k} = \binom{n}{n-k}} \qquad \Big(0 \le k \le n\Big).$$