Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – IX

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§9. Binomial expansions.

I have two goals in mind for this class. One is — this might sound a cliché, but — cover what's taught everywhere in the rest of the world, a subset of the greatest common divisors of requisite math, a 'must-know' for everybody. Think of that as parallel to, say, why the electoral vote system was thought to best serve the country when instituted, why low pressure leads to rain, or what is the wisdom behind Shakespeare's quote "*Love is blind.*" Now, what is actually taught under the name 'math' is so diverse. In our current (North American) system, some math classes (mainly lower division math classes) outstretch towards some eclectic studies, often with tennuous connection with math. (Your course work involves some number-crunching doesn't mean it is math. But in some people's definition, that's still math. What I refer to as 'math' is "math within math", or "insider's math". More below.) Some 'math' textbooks seem to comprise such dispersion, but I am not here to slam a different philosophy. But let me just offer one cautionary tale:

Question. According to one ice cream manufacturer, their ice cream products are stored at retailers at 10°F. You eat their ice cream which is 10°F, and then go outside where the air temperature is 0°F. Then you would feel warmer than if you did not eat that ice cream and went outside. Explain.

I know this is an egregious example. But empirical evidence suggests that most students would identify this problem as a math problem. The correct answer is — save that this can be a legit question in human biology — "there is no correct answer", as a math problem. So, what does this entail? An overdose of 'word problems' (probably designed to achieve some decent goal) has an inadvertent effect of obscuring students' understanding of what math is (and what math is not). Again, I am not going to overgeneralize it and argue against a part of our curricula. But it's always good to be cautiously critical about the presumed merit of the existing practice. So, I rally like 'let's not forget about basic core math'. (That is consistent with the designed complexion of Math 105 as found in our Departmental course catalog, a part of which says "inductive" and "axiomatic" treatment of math.) But I'm not saying that I am not going to use real life metaphors in this class either. I will do it judiciously.

I am in this business long enough to know this is coming. 'Boring', right? Not so fast. Certain knowledge stands the test of time. You might be second-guessing that this class is teaching what's outdated, or obsolete, just because you seem to keep hearing names from the past centuries. Not at all. Remember, there are problems posed centuries ago and have not been solved yet (such as the Riemann Hypothesis). That's pretty common in math. In some science and humanity, if you write up a scholarly article and you cite another article published more than ten years ago then that alone may be an exposure of your skimpy knowledge and low competence. In math, that is not the case at all. It's normal to see first-rate research articles published in the top-notch journals that cite Riemann's paper (the nineteenth century), or Euler's paper (the eighteenth century). That is not because math is a cheap, phony, bogus, subject. Careful, folks. We don't want to just look at how long a certain scientific idea has existed and say it is out-of-date. An idea being succeeded by the next generation without a change could mean that the value of the idea is more permanent. That is the case with math. In other science, theories are frequently refuted, and modified. The same never happens in math. The nature of mathematical knowledge is it is immortal, once built. There is actually another outstanding problem which stood for 350 years, and finally solved in 1995 (Fermat's last theorem). In math, there are problems so easy to state, yet so difficult to solve or even find a clue how to attack it. Fermat's last theorem is one of those. I am going to tell you what that is all about later, but this is a perfect counterexample to the generally held belief, that a math problem that looks easy is actually easy.

'Who cares'? I hear you. Now, before I retort, let me make clear of my second goal. Math is not just drills. If you are a psyche major, you probably got tired of having to react to "oh, so can you read my mind?" joke. For us, "oh, so, such and such number times so and so number (randomly picked large numbers) makes how much?" In people's mind, there is a conception that higher level math is just multiplying large numbers, that math professors don't know any more math than students save some skills to do some such meaningless number-crunching. That's as absurd as saying a psychologist can read others' mind and that's all they do. Most randomly picked large numbers, such as 10^{100}, have mathematically no significance, so mathematicians don't take another look at them. But as we've seen in the last lecture, some large numbers mathematicians do care about, a kind that have bearings on some of the human's ultimate intellectual inquiries. Throughout this semester until now, I never for once said that there is no room to add any new insight to math, or math is a "complete" subject. So my second goal is to help you acclimate to the notion math is a cutting edge science, as in updating of the current state of affairs is occurring literally every hour at every most advanced corners of math. Tens of thousands of preprints (papers that have not yet gone through the peer-review process) are posted in the most publicized math archive on-line servers every day. If you walk out of this room on the last day of the semester, with a firm idea that math is profound, unfathomable and endless, rather than with a rather impassive note with adjectives 'cool', 'uncool', or 'boring', then I call it a 'success'.

There is actually someting that adds another dimension to the 'boredom'. If you say my class is 'boring' (which is what I almost always seem to hear from a certain fraction of students), then the subject matter is boring to you. I'm cool. That's partially me being targetted, in the sense I choose the subject. Perhaps. Yet I'm cool. Sadly, I seem to occasionally have some careless students as in no one can seemingly change their mindset, and I hope that's not you, but know that I am not going to turn my class into an entertainment show either. But if you care enough to be listening right now, it's worth addressing it. Boredom on your part may stem from another peculiar aspect of math, that is, math is 'esoteric'. For example, physicists have the decisive advantage over mathematicians in explaining to non-experts what they do in their forefront research. Say blackholes, or say subatomic particles, people do have some ideas what those are. Mathematicians are naturally handicapped in that department, due to the abhorrently abstract nature of their work. So I have a hard time explaining, when I cover, say binomial coefficients, how does that lead up to a cutting edge forefront science. That is in stark contrast with an astrophysics professor navigating her/his class through sprinkling words like galaxies, gravity, cosmic rays, dark energy, time travel, and the ultimate fate of the universe. You already have the imagination and taste and liking towards the subject, though that's certainly not all of what it takes for you to thrive. But no feeling of boredom means you have already broken your mental inertia to get started and get involved. Math class, on the other hand, is prone to be deemed as boring, because people have no pre-endowed imagination towards the subject. So, I know where you are coming from. But binomial coefficients are so ultra-basic and prevalent in math that there is no way you can avoid it. So, like I said one of my goals is to cover some of those ultra-basic and prevalent concepts in math. At the same time, I am ambitious enough and plan to show you a glimpse of some cutting edge math, a math equivalent of the mystery of blackholes, all while covering some basics everybody needs to know. I am going to snatch some of those from my own (and other) research fields. How does that sound?

- Let's recite Binomial Formula $\big($'Formula B' from "Review of Lectures – VII"$\big)$:

Formula B (Binomial Formula). Let n be a positive integer. Then

$$\boxed{\begin{aligned} &\big(x+y\big)^n \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \binom{n}{3} x^{n-3}y^3 + \cdots \\ &\quad + \binom{n}{n-3} x^3 y^{n-3} + \binom{n}{n-2} x^2 y^{n-2} + \binom{n}{n-1} x\, y^{n-1} + \binom{n}{n} y^n. \end{aligned}}$$

For $n = 1, 2, 3, 4, 5$ and 6, this is

(1) $\big(x+y\big)^1 = \binom{1}{0} x + \binom{1}{1} y,$

(2) $\big(x+y\big)^2 = \binom{2}{0} x^2 + \binom{2}{1} x\,y + \binom{2}{2} y^2,$

(3) $\big(x+y\big)^3 = \binom{3}{0} x^3 + \binom{3}{1} x^2\,y + \binom{3}{2} x\,y^2 + \binom{3}{3} y^3,$

(4) $\big(x+y\big)^4 = \binom{4}{0} x^4 + \binom{4}{1} x^3\,y + \binom{4}{2} x^2\,y^2 + \binom{4}{3} x\,y^3 + \binom{4}{4} y^4,$

(5) $\big(x+y\big)^5$

$$= \binom{5}{0} x^5 + \binom{5}{1} x^4\,y + \binom{5}{2} x^3\,y^2 + \binom{5}{3} x^2\,y^3 + \binom{5}{4} x\,y^4 + \binom{5}{5} y^5,$$

(6) $\big(x+y\big)^6$

$$= \binom{6}{0} x^6 + \binom{6}{1} x^5\,y + \binom{6}{2} x^4\,y^2 + \binom{6}{3} x^3\,y^3 + \binom{6}{4} x^2\,y^4$$
$$+ \binom{6}{5} x\,y^5 + \binom{6}{6} y^6.$$

The coefficients are the binomial coefficients. They are actual numbers. So let's throw the actual numbers:

$$(1) \quad \left(x + y\right)^1 = x + y,$$

$$(2) \quad \left(x + y\right)^2 = x^2 + \underline{\underline{2}}\, x\, y + y^2,$$

$$(3) \quad \left(x + y\right)^3 = x^3 + \underline{\underline{3}}\, x^2\, y + \underline{\underline{3}}\, x\, y^2 + y^3,$$

$$(4) \quad \left(x + y\right)^4 = x^4 + \underline{\underline{4}}\, x^3\, y + \underline{\underline{6}}\, x^2\, y^2 + \underline{\underline{4}}\, x\, y^3 + y^4,$$

$$(5) \quad \left(x + y\right)^5 = x^5 + \underline{\underline{5}}\, x^4\, y + \underline{\underline{10}}\, x^3\, y^2 + \underline{\underline{10}}\, x^2\, y^3 + \underline{\underline{5}}\, x\, y^4 + y^5,$$

$$(6) \quad \left(x + y\right)^6 = x^6 + \underline{\underline{6}}\, x^5\, y + \underline{\underline{15}}\, x^4\, y^2 + \underline{\underline{20}}\, x^3\, y^3 + \underline{\underline{15}}\, x^2\, y^4 + \underline{\underline{6}}\, x\, y^5 + y^6,$$

Let's substitute concrete numbers for y in the above (1) through (6).

• $\boldsymbol{y = 1.}$ This is easy. So, for example, as a result of substituting $y = 1$ in $10\,x^3\,y^2$, you will get $10\,x^3 \cdot 1^2$, but this is just $10\,x^3$. Also, as a result of substituting $y = 1$ in y^6, you will get 1^6, but this is just 1. And so on so forth. In short, y-to-the-powers are all replaced with 1. Accordingly,

$$(1) \quad \left(x + 1\right)^1 = x + 1,$$

$$(2) \quad \left(x + 1\right)^2 = x^2 + 2\,x + 1,$$

$$(3) \quad \left(x + 1\right)^3 = x^3 + 3\,x^2 + 3\,x + 1,$$

$$(4) \quad \left(x + 1\right)^4 = x^4 + 4\,x^3 + 6\,x^2 + 4\,x + 1,$$

$$(5) \quad \left(x + 1\right)^5 = x^5 + 5\,x^4 + 10\,x^3 + 10\,x^2 + 5\,x + 1,$$

$$(6) \quad \left(x + 1\right)^6 = x^6 + 6\,x^5 + 15\,x^4 + 20\,x^3 + 15\,x^2 + 6\,x + 1.$$

- $\boldsymbol{y = 2.}$

$$(1) \qquad \Big(x + 2\Big)^1 = x + 2,$$

$$
\begin{aligned}
(2) \qquad \Big(x + 2\Big)^2 &= x^2 + 2 \cdot x \cdot 2 + 2^2 \\
&= x^2 + 4\,x + 4,
\end{aligned}
$$

$$
\begin{aligned}
(3) \qquad \Big(x + 2\Big)^3 &= x^3 + 3 \cdot x^2 \cdot 2 + 3 \cdot x \cdot 2^2 + 2^3 \\
&= x^3 + 6\,x^2 + 12\,x + 8,
\end{aligned}
$$

$$
\begin{aligned}
(4) \qquad \Big(x + 2\Big)^4 &= x^4 + 4 \cdot x^3 \cdot 2 + 6 \cdot x^2 \cdot 2^2 + 4 \cdot x \cdot 2^3 + 2^4 \\
&= x^4 + 8\,x^3 + 24\,x^2 + 32\,x + 16,
\end{aligned}
$$

$$
\begin{aligned}
(5) \qquad \Big(x + 2\Big)^5 &= x^5 + 5 \cdot x^4 \cdot 2 + 10 \cdot x^3 \cdot 2^2 + 10 \cdot x^2 \cdot 2^3 + 5 \cdot x \cdot 2^4 + 2^5 \\
&= x^5 + 10\,x^4 + 40\,x^3 + 80\,x^2 + 80\,x + 32,
\end{aligned}
$$

$$
\begin{aligned}
(6) \qquad \Big(x + 2\Big)^6 &= x^6 + 6 \cdot x^4 \cdot 2 + 15 \cdot x^4 \cdot 2^2 + 20 \cdot x^3 \cdot 2^3 + 15 \cdot x^2 \cdot 2^4 \\
&\qquad + 6 \cdot x \cdot 2^5 + 2^6 \\
&= x^6 + 12\,x^5 + 60\,x^4 + 160\,x^3 + 240\,x^2 + 192\,x + 64.
\end{aligned}
$$

- $\boldsymbol{y = 3.}$

$$(1) \qquad \Big(x + 3\Big)^1 = x + 3,$$

$$(2) \qquad \Big(x + 3\Big)^2 = x^2 + 2 \cdot x \cdot 3 + 3^2$$

$$= x^2 + 6\,x + 9,$$

$$(3) \qquad \Big(x + 3\Big)^3 = x^3 + 3 \cdot x^2 \cdot 3 + 3 \cdot x \cdot 3^2 + 3^3$$

$$= x^3 + 9\,x^2 + 27\,x + 27,$$

$$(4) \qquad \Big(x + 3\Big)^4 = x^4 + 4 \cdot x^3 \cdot 3 + 6 \cdot x^2 \cdot 3^2 + 4 \cdot x \cdot 3^3 + 3^4$$

$$= x^4 + 12\,x^3 + 54\,x^2 + 108\,x + 81,$$

$$(5) \qquad \Big(x + 3\Big)^5 = x^5 + 5 \cdot x^4 \cdot 3 + 10 \cdot x^3 \cdot 3^2 + 10 \cdot x^2 \cdot 3^3 + 5 \cdot x \cdot 3^4 + 3^5$$

$$= x^5 + 15\,x^4 + 90\,x^3 + 270\,x^2 + 405\,x + 243,$$

$$(6) \qquad \Big(x + 3\Big)^6 = x^6 + 6 \cdot x^4 \cdot 3 + 15 \cdot x^4 \cdot 3^2 + 20 \cdot x^3 \cdot 3^3 + 15 \cdot x^2 \cdot 3^4$$

$$+ 6 \cdot x \cdot 3^5 + 3^6$$

$$= x^6 + 18\,x^5 + 135\,x^4 + 540\,x^3 + 1215\,x^2 + 1458\,x + 729.$$

- **Summary.**

$$(1) \quad \Big(x+1\Big)^1 = x + 1,$$
$$(2) \quad \Big(x+1\Big)^2 = x^2 + 2x + 1,$$
$$(3) \quad \Big(x+1\Big)^3 = x^3 + 3x^2 + 3x + 1,$$
$$(4) \quad \Big(x+1\Big)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1,$$
$$(5) \quad \Big(x+1\Big)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1,$$
$$(6) \quad \Big(x+1\Big)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1.$$

$$(1)' \quad \Big(x+2\Big)^1 = x + 2,$$
$$(2)' \quad \Big(x+2\Big)^2 = x^2 + 4x + 4,$$
$$(3)' \quad \Big(x+2\Big)^3 = x^3 + 6x^2 + 12x + 8,$$
$$(4)' \quad \Big(x+2\Big)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16,$$
$$(5)' \quad \Big(x+2\Big)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32,$$
$$(6)' \quad \Big(x+2\Big)^6 = x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64.$$

$$(1)'' \quad \Big(x+3\Big)^1 = x + 3,$$
$$(2)'' \quad \Big(x+3\Big)^2 = x^2 + 6x + 9,$$
$$(3)'' \quad \Big(x+3\Big)^3 = x^3 + 9x^2 + 27x + 27,$$
$$(4)'' \quad \Big(x+3\Big)^4 = x^4 + 12x^3 + 54x^2 + 108x + 81,$$
$$(5)'' \quad \Big(x+3\Big)^5 = x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243,$$
$$(6)'' \quad \Big(x+3\Big)^6 = x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x + 729.$$

• More generally, you can substitute any number for y $\big($or $x\big)$ and obtain different expansion formulas. In the above, we have substituted $y = 1,\ 2$ and 3. But you may substitute $y = -1,\ -2,\ -3,$ *etc.* as well. In that case, What you will end up getting is a formula for

$$\begin{array}{cccccc}
\Big(x-1\Big)^1, & \Big(x-1\Big)^2, & \Big(x-1\Big)^3, & \Big(x-1\Big)^4, & \Big(x-1\Big)^5, & \Big(x-1\Big)^6, \\
\Big(x-2\Big)^1, & \Big(x-2\Big)^2, & \Big(x-2\Big)^3, & \Big(x-2\Big)^4, & \Big(x-2\Big)^5, & \Big(x-2\Big)^6, \\
\Big(x-3\Big)^1, & \Big(x-3\Big)^2, & \Big(x-3\Big)^3, & \Big(x-3\Big)^4, & \Big(x-3\Big)^5, & \Big(x-3\Big)^6, \\
\vdots & & \vdots & & & \vdots
\end{array}$$

During that process, you are going to have to deal with

$$\begin{array}{cccccc}
\Big(-1\Big)^1, & \Big(-1\Big)^2, & \Big(-1\Big)^3, & \Big(-1\Big)^4, & \Big(-1\Big)^5, & \Big(-1\Big)^6, \\
\Big(-2\Big)^1, & \Big(-2\Big)^2, & \Big(-2\Big)^3, & \Big(-2\Big)^4, & \Big(-2\Big)^5, & \Big(-2\Big)^6, \\
\Big(-3\Big)^1, & \Big(-3\Big)^2, & \Big(-3\Big)^3, & \Big(-3\Big)^4, & \Big(-3\Big)^5, & \Big(-3\Big)^6, \\
\vdots & & \vdots & & & \vdots
\end{array}$$

As for this, remember that we have covered

$$\begin{array}{rcr}
\Big(-a\Big)^1 & = & -\,a, \\
\Big(-a\Big)^2 & = & a^2, \\
\Big(-a\Big)^3 & = & -\,a^3, \\
\Big(-a\Big)^4 & = & a^4, \\
\Big(-a\Big)^5 & = & -\,a^5, \\
\Big(-a\Big)^6 & = & a^6, \\
\vdots & & \vdots
\end{array}$$

So

$$
\begin{aligned}
\Big(-1\Big)^1 &= -1, \\
\Big(-1\Big)^2 &= 1^2, \\
\Big(-1\Big)^3 &= -1^3, \\
\Big(-1\Big)^4 &= 1^4, \\
\Big(-1\Big)^5 &= -1^5, \\
\Big(-1\Big)^6 &= 1^6, \\
\\
\Big(-2\Big)^1 &= -2, \\
\Big(-2\Big)^2 &= 2^2, \\
\Big(-2\Big)^3 &= -2^3, \\
\Big(-2\Big)^4 &= 2^4, \\
\Big(-2\Big)^5 &= -2^5, \\
\Big(-2\Big)^6 &= 2^6, \\
\\
\Big(-3\Big)^1 &= -3, \\
\Big(-3\Big)^2 &= 3^2, \\
\Big(-3\Big)^3 &= -3^3, \\
\Big(-3\Big)^4 &= 3^4, \\
\Big(-3\Big)^5 &= -3^5, \\
\Big(-3\Big)^6 &= 3^6, \\
\vdots \quad & \quad \vdots
\end{aligned}
$$

Taking these into account, we may substitute $y = -1$ into each of

$$(1) \quad \left(x + y\right)^1 = x + y,$$

$$(2) \quad \left(x + y\right)^2 = x^2 + \underline{\underline{2}}\, x\, y + y^2,$$

$$(3) \quad \left(x + y\right)^3 = x^3 + \underline{\underline{3}}\, x^2 y + \underline{\underline{3}}\, x\, y^2 + y^3,$$

$$(4) \quad \left(x + y\right)^4 = x^4 + \underline{\underline{4}}\, x^3 y + \underline{\underline{6}}\, x^2 y^2 + \underline{\underline{4}}\, x\, y^3 + y^4,$$

$$(5) \quad \left(x + y\right)^5 = x^5 + \underline{\underline{5}}\, x^4 y + \underline{\underline{10}}\, x^3 y^2 + \underline{\underline{10}}\, x^2 y^3 + \underline{\underline{5}}\, x\, y^4 + y^5,$$

$$(6) \quad \left(x + y\right)^6 = x^6 + \underline{\underline{6}}\, x^5 y + \underline{\underline{15}}\, x^4 y^2 + \underline{\underline{20}}\, x^3 y^3 + \underline{\underline{15}}\, x^2 y^4 + \underline{\underline{6}}\, x\, y^5 + y^6,$$

and as a result, we get

- $\boldsymbol{y = -1.}$

$$(1) \quad \left(x - 1\right)^1 = x - 1,$$

$$(2) \quad \left(x - 1\right)^2 = x^2 - 2x + 1,$$

$$(3) \quad \left(x - 1\right)^3 = x^3 - 3x^2 + 3x - 1,$$

$$(4) \quad \left(x - 1\right)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1,$$

$$(5) \quad \left(x - 1\right)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1,$$

$$(6) \quad \left(x - 1\right)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1.$$

Notice that the sign on every other terms is negated.

Similarly, we may substitute $y = -2$ and $y = -3$ each into the above:

- $\boldsymbol{y = -2.}$

$$(1)' \quad \left(x - 2\right)^1 = x - 2,$$
$$(2)' \quad \left(x - 2\right)^2 = x^2 - 4x + 4,$$
$$(3)' \quad \left(x - 2\right)^3 = x^3 - 6x^2 + 12x - 8,$$
$$(4)' \quad \left(x - 2\right)^4 = x^4 - 8x^3 + 24x^2 - 32x + 16,$$
$$(5)' \quad \left(x - 2\right)^5 = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32,$$
$$(6)' \quad \left(x - 2\right)^6 = x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64.$$

- $\boldsymbol{y = -3.}$

$$(1)'' \quad \left(x - 3\right)^1 = x - 3,$$
$$(2)'' \quad \left(x - 3\right)^2 = x^2 - 6x + 9,$$
$$(3)'' \quad \left(x - 3\right)^3 = x^3 - 9x^2 + 27x - 27,$$
$$(4)'' \quad \left(x - 3\right)^4 = x^4 - 12x^3 + 54x^2 - 108x + 81,$$
$$(5)'' \quad \left(x - 3\right)^5 = x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243,$$
$$(6)'' \quad \left(x - 3\right)^6 = x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729.$$

Exercise 1. Expand each of

$$\left(x + 6\right)^2. \qquad \left(x + 5\right)^3. \qquad \left(x + 4\right)^4. \qquad \left(x + 1\right)^7.$$

$$\left(x - 6\right)^2. \qquad \left(x - 5\right)^3. \qquad \left(x - 4\right)^4. \qquad \left(x - 1\right)^7.$$

$\left[\underline{\textbf{Answers}}\right]$:

$$\left(x + 6\right)^2 = x^2 + 12\,x + 36,$$
$$\left(x - 6\right)^2 = x^2 - 12\,x + 36,$$

$$\left(x + 5\right)^3 = x^3 + 15\,x^2 + 75\,x + 125,$$
$$\left(x - 5\right)^3 = x^3 - 15\,x^2 + 75\,x - 125,$$

$$\left(x + 4\right)^4 = x^4 + 16\,x^3 + 96\,x^2 + 256\,x + 256,$$
$$\left(x - 4\right)^4 = x^4 - 16\,x^3 + 96\,x^2 - 256\,x + 256,$$

$$\left(x + 1\right)^7 = x^7 + 7\,x^6 + 21\,x^5 + 35\,x^4 + 35\,x^3 + 21\,x^2 + 7\,x + 1,$$
$$\left(x - 1\right)^7 = x^7 - 7\,x^6 + 21\,x^5 - 35\,x^4 + 35\,x^3 - 21\,x^2 + 7\,x - 1.$$