Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – VII

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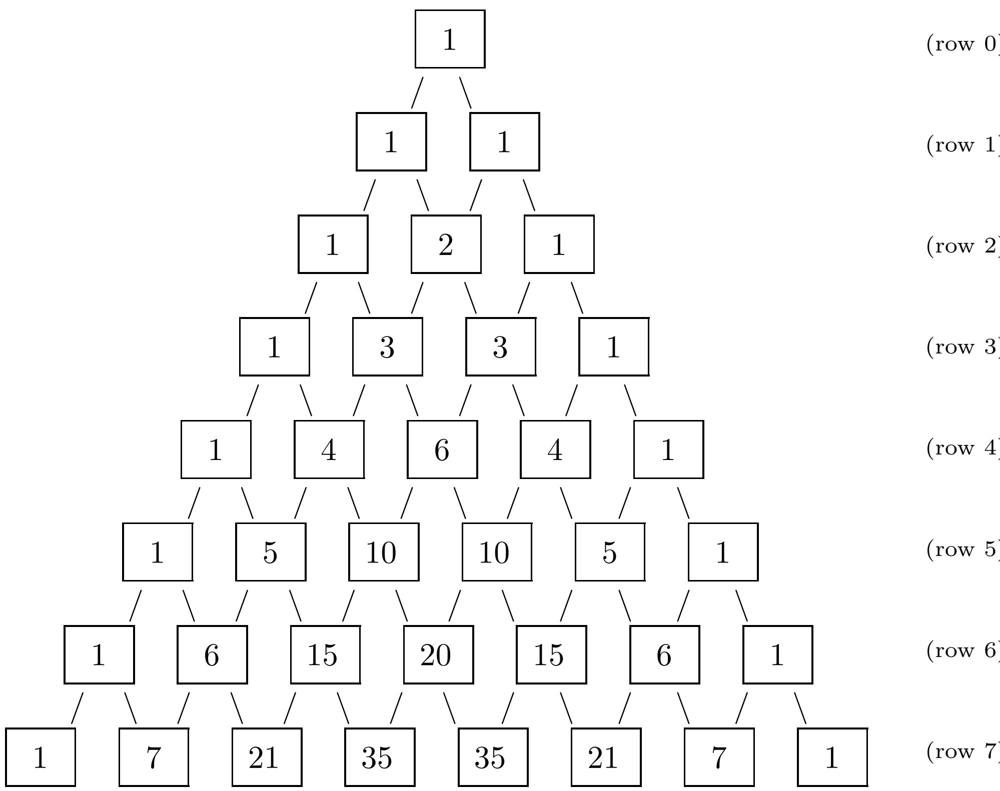
§7. Binomial formula.

Three lectures ago (in "Review of Lectuires – IV"), we have covered



This is called the <u>Pascal's triangle</u>. How the numbers are arranged in the Pascal's triangle is dictated by the rule:

Rule.

<u>At every spot, that number equals the sum of two numbers right above it.</u>

Meanwhile, let's recall two formulas

$$\boxed{\left(x + y\right)^2 \;=\; x^2 \;+\; 2\,x\,y \;+\; y^2}\,,$$

$$\boxed{\left(x + y\right)^3 \;=\; x^3 \;+\; 3\,x^2\,y \;+\; 3\,x\,y^2 \;+\; y^3}\,.$$

(The first one from "Review of Lectures – V", the second one from "Review of Lectures – VI".) Today I want to bridge these two subjects. As a starter, let's address <u>what comes next to squaring and cubing</u>.

- **Higher powers.** We have defined x^2 and x^3:

$$\boxed{x^2 \;=\; x \cdot x}\,, \qquad \text{and} \qquad \boxed{x^3 \;=\; x \cdot x \cdot x}\,.$$

(the <u>square</u> of x) (the <u>cube</u> of x)

It is very natural to extend this and consider

$$\begin{aligned}
x^0 &= 1, \\
x^1 &= x, \\
x^2 &= x \cdot x, \\
x^3 &= x \cdot x \cdot x. \\
x^4 &= x \cdot x \cdot x \cdot x, \\
x^5 &= x \cdot x \cdot x \cdot x \cdot x, \\
x^6 &= x \cdot x \cdot x \cdot x \cdot x \cdot x, \\
x^7 &= x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x, \\
&\vdots \qquad\qquad\qquad\qquad \ddots
\end{aligned}$$

So here is the formal definition:

Definition. For a positive integer n, define x^n as

$$\boxed{x^n = \underbrace{x \cdot x \cdot x \cdot \quad \cdots \quad \cdot x}_{n}.}$$

This is pronounced as

“ $\underline{\underline{x \text{ to the } n\text{-th power}}}$ ”,

“ $\underline{\underline{x \text{ raised to the power of } n}}$ ”,

or simply

“ $\underline{\underline{x \text{ to the } n}}$ ”.

Example 1. For $n = 24$,

$$\begin{aligned} x^{24} = \quad & x \cdot x \cdot x \cdot x \cdot x \cdot x \\ & \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \\ & \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \\ & \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \end{aligned}$$

(“x to the 24”).

⋆ Let me repeat that, for $n = 0$ and $n = 1$, we set

$$\boxed{x^0 = 1} \qquad \text{and}$$

$$\boxed{x^1 = x},$$

and these are $\underline{\text{by convention}}$.

Example 2. $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81.$

Example 3. $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243.$

Example 4. $4^4 = 4 \cdot 4 \cdot 4 \cdot 4 = 256.$

Example 5. $5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625.$

Example 6. $3^6 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 729.$

Example 7. $4^5 = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 1024.$

Example 8. $6^4 = 6 \cdot 6 \cdot 6 \cdot 6 = 1296.$

Example 9. $7^4 = 7 \cdot 7 \cdot 7 \cdot 7 = 2401.$

Example 10. $5^5 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 3125.$

Example 11. $4^6 = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4096.$

Example 12.

$$10^4 = 10 \cdot 10 \cdot 10 \cdot 10 = 10000 \ \big(\text{ten thousand}\big).$$
$$10^5 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100000 \ \big(\text{one hundred thousand}\big).$$
$$10^6 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1000000 \ \big(\text{one million}\big).$$
$$10^7 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000000 \ \big(\text{ten million}\big).$$
$$10^8 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100000000 \ \big(\text{one hundred million}\big).$$
$$10^9 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1000000000 \ \big(\text{one billion}\big).$$

○ More generally, for a positive integer n,

$$\boxed{10^n = 1\ \underbrace{0\,0\,0\,0\,0\ \ldots\ 0}_{n}}\ .$$

⋆ In mathematics, we don't place ' , ' (comma) after every third digit.

- **0-to-the-powers.** We have

$$
\begin{aligned}
0^1 &= 0,\\
0^2 &= 0,\\
0^3 &= 0,\\
0^4 &= 0,\\
0^5 &= 0,\\
0^6 &= 0,\\
0^7 &= 0,\\
\vdots & \quad\ \vdots
\end{aligned}
$$

— Well, nothing impressive or dramatic.

- **1-to-the-powers.** We have

$$
\begin{aligned}
1^1 &= 1,\\
1^2 &= 1,\\
1^3 &= 1,\\
1^4 &= 1,\\
1^5 &= 1,\\
1^6 &= 1,\\
1^7 &= 1,\\
\vdots & \quad\ \vdots
\end{aligned}
$$

— Again, nothing impressive or dramatic.

By the way,

$$\boxed{1^0 = 1} \quad \text{whereas} \quad \underline{\underline{\boxed{0^0}}} \ \ \underline{\underline{\text{is undefined}}}\ .$$

- **$\left(-1\right)$-to-the-powers.** We have

$$\begin{aligned}
\left(-1\right)^{1} &= -1, \\
\left(-1\right)^{2} &= 1, \\
\left(-1\right)^{3} &= -1, \\
\left(-1\right)^{4} &= 1, \\
\left(-1\right)^{5} &= -1, \\
\left(-1\right)^{6} &= 1, \\
\left(-1\right)^{7} &= -1, \\
\left(-1\right)^{8} &= 1, \\
\left(-1\right)^{9} &= -1, \\
\left(-1\right)^{10} &= 1, \\
&\vdots
\end{aligned}$$

In short,

$$\boxed{\left(-1\right)^{n} = \begin{cases} 1 & \left(\text{if } n \text{ is } \underline{\underline{\text{even}}}\right), \\ -1 & \left(\text{if } n \text{ is } \underline{\underline{\text{odd}}}\right). \end{cases}}$$

Exercise 1. Find each of

$$\left(-1\right)^{15}, \qquad \left(-1\right)^{48}, \qquad \left(-1\right)^{91}.$$

[**Answers**]: $\left(-1\right)^{15} = -1.$ $\left(-1\right)^{48} = 1.$ $\left(-1\right)^{91} = -1.$

- $\left(-\boldsymbol{a}\right)$**-to-the-powers.** We have

$$\begin{aligned}
\left(-a\right)^1 &= -a,\\
\left(-a\right)^2 &= a^2,\\
\left(-a\right)^3 &= -a^3,\\
\left(-a\right)^4 &= a^4,\\
\left(-a\right)^5 &= -a^5,\\
\left(-a\right)^6 &= a^6,\\
\left(-a\right)^7 &= -a^7,\\
\left(-a\right)^8 &= a^8,\\
\left(-a\right)^9 &= -a^9,\\
\left(-a\right)^{10} &= a^{10},\\
&\vdots
\end{aligned}$$

In short,

$$\boxed{\left(-a\right)^n = \begin{cases} a^n & \left(\text{if } n \text{ is } \underline{\underline{\text{even}}}\right), \\ -a^n & \left(\text{if } n \text{ is } \underline{\underline{\text{odd}}}\right). \end{cases}}$$

Exercise 2. Find each of

$$\left(-2\right)^6, \qquad \left(-3\right)^5, \qquad \left(-5\right)^4.$$

[**Answers**]**:** $\left(-2\right)^6 = 64.$ $\left(-3\right)^5 = -243.$ $\left(-5\right)^4 = 625.$

- **2-to-the-powers.**

The numbers in the following sequence are called "<u>2-to-the-powers</u>":

$$
\begin{aligned}
2^1 &= 2,\\
2^2 &= 4,\\
2^3 &= 8,\\
2^4 &= 16,\\
2^5 &= 32,\\
2^6 &= 64,\\
2^7 &= 128,\\
2^8 &= 256,\\
2^9 &= 512,\\
2^{10} &= 1024,\\
2^{11} &= 2048,\\
2^{12} &= 4096,\\
2^{13} &= 8192,\\
2^{14} &= 16384,\\
2^{15} &= 32768,\\
2^{16} &= 65536,\\
\vdots \quad & \quad \vdots
\end{aligned}
$$

2-to-the-powers frequently appear in mathematics. Please familiarize yourself with the above listed numbers (the first sixteen of 2-to-the-powers).

Exercise 3. Identify all 2-to-the-powers among the numbers listed below. Write each of those 2-to-the-powers in the form 2^n with a concrete positive integer n.

8,	12,	24,	32,	48,	64,	80,	84,	128,
144,	216,	256,	360,	384,	480,	512,	768,	784,
1296,	1440,	2016,	2048,	2560,	3840,	5040,	6912,	8192.

Answer: 8, 32, 64, 128, 256, 512, 2048, 8192.

$$8 = 2^3, \qquad 32 = 2^5, \qquad 64 = 2^6, \qquad 128 = 2^7,$$
$$256 = 2^8, \qquad 512 = 2^9, \qquad 2048 = 2^{11}, \qquad 8192 = 2^{13}.$$

- **Binomial Formula.**

Finally today's main theme. Check this out:

$$(1) \quad \Big(x + y\Big)^1 = x + y,$$
$$(2) \quad \Big(x + y\Big)^2 = x^2 + \underline{\underline{2}}\, x\, y + y^2,$$
$$(3) \quad \Big(x + y\Big)^3 = x^3 + \underline{\underline{3}}\, x^2\, y + \underline{\underline{3}}\, x\, y^2 + y^3,$$
$$(4) \quad \Big(x + y\Big)^4 = x^4 + \underline{\underline{4}}\, x^3\, y + \underline{\underline{6}}\, x^2\, y^2 + \underline{\underline{4}}\, x\, y^3 + y^4,$$
$$(5) \quad \Big(x + y\Big)^5 = x^5 + \underline{\underline{5}}\, x^4\, y + \underline{\underline{10}}\, x^3\, y^2 + \underline{\underline{10}}\, x^2\, y^3 + \underline{\underline{5}}\, x\, y^4 + y^5,$$
$$(6) \quad \Big(x + y\Big)^6 = x^6 + \underline{\underline{6}}\, x^5\, y + \underline{\underline{15}}\, x^4\, y^2 + \underline{\underline{20}}\, x^3\, y^3 + \underline{\underline{15}}\, x^2\, y^4 + \underline{\underline{6}}\, x\, y^5 + y^6,$$
$$\vdots \qquad \qquad \qquad \ddots$$

The first one is just a tautology. The next two we have covered in our last two lectures. Others look new to you. Let's dissect. First, we have a convenient word for the underlined numbers in the above. They are called the <u>coefficients</u>. Also, for example, x^3 can be regarded as an abbreviation for $1\, x^3$, *etc.* So we say that, in the right-hand side of each of the above lines, the first and the last terms both have coefficient 1. Thus:

- The <u>coefficient</u> of $x^3\, y$ in the right-hand side of (4) is 4.
- The <u>coefficient</u> of $x^2\, y^3$ in the right-hand side of (5) is 10.
- The <u>coefficient</u> of y^6 in the right-hand side of (6) is 1.

Now, those underlined numbers $\big($coefficients$\big)$ look familiar. If we just pick up those coefficients from left to right, in each line:

(1)	1, 1.
(2)	1, 2, 1.
(3)	1, 3, 3, 1.
(4)	1, 4, 6, 4, 1.
(5)	1, 5, 10, 10, 5, 1.
(6)	1, 6, 15, 20, 15, 6, 1.

Yes indeed: This is exactly the Pascal's triangle. So, can you guess the formula for $\big(x+y\big)^7$ and $\big(x+y\big)^8$ each? Yes, according to Pascal, the list of coefficients continues as

(7)	1, 7, 21, 35, 35, 21, 7, 1.
(8)	1, 8, 28, 56, 70, 56, 28, 8, 1.

Accordingly,

(7) $\big(x + y\big)^7$

$$= x^7 + \underline{\underline{7}}\,x^6 y + \underline{\underline{21}}\,x^5 y^2 + \underline{\underline{35}}\,x^4 y^3 + \underline{\underline{35}}\,x^3 y^4 + \underline{\underline{21}}\,x^2 y^5 + \underline{\underline{7}}\,x y^6 + y^7.$$

(8) $\big(x + y\big)^8$

$$= x^8 + \underline{\underline{8}}\,x^7 y + \underline{\underline{28}}\,x^6 y^2 + \underline{\underline{56}}\,x^5 y^3 + \underline{\underline{70}}\,x^4 y^4 + \underline{\underline{56}}\,x^3 y^5 + \underline{\underline{35}}\,x^2 y^6 + \underline{\underline{8}}\,x y^7 + y^8.$$

These are indeed the correct formulas. Now, at this point I'm sure you already know how to form the correct formula for $\big(x+y\big)^n$ for $n = 9, 10, \ldots$. I want to officially formulate it. That requires me to introduce one new notation.

Notation (binomial coefficient). In the Pascal's triangle (page 1):

- The numbers in row 1, from left to right, are denoted as

$$\begin{matrix} \binom{1}{0}, & \binom{1}{1}. \\ \| & \| \\ 1 & 1 \end{matrix}$$

- The numbers in row 2, from left to right, are denoted as

$$\begin{matrix} \binom{2}{0}, & \binom{2}{1}, & \binom{2}{2}. \\ \| & \| & \| \\ 1 & 2 & 1 \end{matrix}$$

- The numbers in row 3, from left to right, are denoted as

$$\begin{matrix} \binom{3}{0}, & \binom{3}{1}, & \binom{3}{2}, & \binom{3}{3}. \\ \| & \| & \| & \| \\ 1 & 3 & 3 & 1 \end{matrix}$$

- The numbers in row 4, from left to right, are denoted as

$$\begin{matrix} \binom{4}{0}, & \binom{4}{1}, & \binom{4}{2}, & \binom{4}{3}, & \binom{4}{4}. \\ \| & \| & \| & \| & \| \\ 1 & 4 & 6 & 4 & 1 \end{matrix}$$

- The numbers in row 5, from left to right, are denoted as

$$\begin{matrix} \binom{5}{0}, & \binom{5}{1}, & \binom{5}{2}, & \binom{5}{3}, & \binom{5}{4}, & \binom{5}{5}. \\ \| & \| & \| & \| & \| & \| \\ 1 & 5 & 10 & 10 & 5 & 1 \end{matrix}$$

$$\vdots \qquad\qquad\qquad\qquad \ddots$$

More generally, in the Pascal:

○ The numbers in row n, from left to right, are denoted as

$$\binom{n}{0}, \quad \binom{n}{1}, \quad \binom{n}{2}, \quad \binom{n}{3}, \quad \cdots \quad \binom{n}{n}.$$

These are called <u>the binomial coefficients</u> . Note

$$\boxed{\binom{n}{0} = 1,} \qquad \boxed{\binom{n}{n} = 1.}$$

Example 13. $\binom{6}{0} = 1.$ $\binom{6}{1} = 6.$ $\binom{6}{3} = 20.$

$$\binom{7}{2} = 21. \quad \binom{7}{5} = 21. \quad \binom{8}{4} = 70. \quad \binom{8}{7} = 8.$$

⋆ As for the formula for $\binom{n}{k}$ for general n and k, see 'Formula A' in the next page.

- Using this new notation, we can rewrite the previous formulas as

(1) $\big(x + y\big)^1 = \binom{1}{0} x + \binom{1}{1} y,$

(2) $\big(x + y\big)^2 = \binom{2}{0} x^2 + \binom{2}{1} x\, y + \binom{2}{2} y^2,$

(3) $\big(x + y\big)^3 = \binom{3}{0} x^3 + \binom{3}{1} x^2\, y + \binom{3}{2} x\, y^2 + \binom{3}{3} y^3,$

(4) $\big(x + y\big)^4 = \binom{4}{0} x^4 + \binom{4}{1} x^3\, y + \binom{4}{2} x^2\, y^2 + \binom{4}{3} x\, y^3 + \binom{4}{4} y^4,$

$\vdots \qquad\qquad \ddots$

Do you see the patterns? More generally, the right formula for $\big(x+y\big)^n$ is given below (Formula B in the next page).

Formula A (binomial coefficients).

Let n and k be integers, with $0 < k < n$. Then

$$\boxed{\binom{n}{k} = \frac{n\,(n-1)\,(n-2)\,\cdots\,(n-k+1)}{1\cdot 2\cdot 3\cdot\cdots\cdot k}}\,.$$

Formula B (Binomial Formula). Let n be a positive integer. Then

$$\boxed{\begin{aligned}&(x+y)^n\\ &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots\\ &\quad + \binom{n}{n-3}x^3y^{n-3} + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}x\,y^{n-1} + \binom{n}{n}y^n.\end{aligned}}$$

Exercise 4. Spell out each of the following binomial coefficients, in the fraction form. You don't have to calculate the answers.

$$\binom{9}{5},\quad \binom{10}{7},\quad \binom{11}{6},\quad \binom{14}{2},\quad \binom{15}{0},\quad \binom{18}{11}.$$

[**Answers**]: $\binom{9}{5} = \dfrac{9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5}$, $\binom{10}{7} = \dfrac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$,

$\binom{11}{6} = \dfrac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$. $\binom{14}{2} = \dfrac{14\cdot 13}{1\cdot 2}$.

$\binom{15}{0} = \dfrac{1}{1}$. $\binom{18}{11} = \dfrac{18\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}$.

Exercise 5. Spell out the binomial formula for each of

(a) $\left(x+y\right)^5$, (b) $\left(x+y\right)^8$, and (c) $\left(x+y\right)^9$.

In each of (a), (b), (c), first give the formula that includes the notation $\binom{n}{k}$. Then convert those $\binom{n}{k}$ into numbers and rewrite your answer accordingly.

[**Answers**]: (a) $\left(x+y\right)^5$

$$= \binom{5}{0}x^5 + \binom{5}{1}x^4 y + \binom{5}{2}x^3 y^2 + \binom{5}{3}x^2 y^3 + \binom{5}{4}x y^4 + \binom{5}{5}y^5$$

$$= x^5 + 5\,x^4 y + 10\,x^3 y^2 + 10\,x^2 y^3 + 5\,x y^4 + y^5,$$

(b) $\left(x+y\right)^8$

$$= \binom{8}{0}x^8 + \binom{8}{1}x^7 y + \binom{8}{2}x^6 y^2 + \binom{8}{3}x^5 y^3 + \binom{8}{4}x^4 y^4$$

$$+ \binom{8}{5}x^3 y^5 + \binom{8}{6}x^2 y^6 + \binom{8}{7}x y^7 + \binom{8}{8}y^8$$

$$= x^8 + 8\,x^7 y + 28\,x^6 y^2 + 56\,x^5 y^3 + 70\,x^4 y^4$$

$$+ 56\,x^3 y^5 + 28\,x^2 y^6 + 8\,x y^7 + y^8.$$

(c) $\left(x+y\right)^9$

$$= \binom{9}{0}x^9 + \binom{9}{1}x^8 y + \binom{9}{2}x^7 y^2 + \binom{9}{3}x^6 y^3 + \binom{9}{4}x^5 y^4$$

$$+ \binom{9}{5}x^4 y^5 + \binom{9}{6}x^3 y^6 + \binom{9}{7}x^2 y^7 + \binom{9}{8}x y^8 + \binom{9}{9}y^9$$

$$= x^9 + 9\,x^8 y + 36\,x^7 y^2 + 84\,x^6 y^3 + 126\,x^5 y^4$$

$$+ 126\,x^4 y^5 + 84\,x^3 y^6 + 36\,x^2 y^7 + 9\,x y^8 + y^9.$$

- **Pop quiz.**

How much does it make it you add up the numbers in one whole row in Pascal?

Let's experiment:

Row 1: $1 + 1 = 2.$

Row 2: $1 + 2 + 1 = 4.$

Row 3: $1 + 3 + 3 + 1 = 8.$

Row 4: $1 + 4 + 6 + 4 + 1 = 16.$

Row 5: $1 + 5 + 10 + 10 + 5 + 1 = 32.$

Row 6: $1 + 6 + 15 + 20 + 15 + 6 + 1 = 64.$

Row 7: $1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128.$

To align the answers:

$$2, \quad 4, \quad 8, \quad 16, \quad 32, \quad 64, \quad 128, \quad \cdots .$$

So, these numbers look familiar, right? Yes, they are 2-to-the-powers. More precisely:

Fact. The sum of the numbers in the n-th row of Pascal equals 2^n.

What I mean by this is that, it is not just for Row 1—Row 7, but if you do the same for the lower rows, the same is always true. But why? The clue is, this is a simple application of the Binomial Formula (Formula B above). I will leave it as your own exercise to figure it out. If you need further clue: Substituting some appropriate number for each of x and y in the Binomial Formula would do it. So, figure out those numbers to be substituted for each of x and y.

Exercise 6. Explain why the following fact follows from the Binomial Theorem:

Fact. The sum of the numbers in the n-th row of Pascal equals 2^n .

Indicate what number to substitute for each of x and y in the Binomial Formula.

[**Answer**]: Substituting $\boxed{x = 1}$ and $\boxed{y = 1}$ in the Binomial Formula

yields

$$\begin{aligned}(1+1)^n &= \binom{n}{0} \cdot 1^n + \binom{n}{1} \cdot 1^{n-1} \cdot 1^1 + \binom{n}{2} \cdot 1^{n-2} \cdot 1^2 + \cdots \\ &\quad + \binom{n}{n-2} \cdot 1^2 \cdot 1^{n-2} + \binom{n}{n-1} \cdot 1^1 \cdot 1^{n-1} + \binom{n}{n} \cdot 1^n \\ &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n}.\end{aligned}$$

In short,

$$\boxed{2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n}} \; .$$