Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – VI

February 2 (Mon), 2015

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§6. Cubes.

- We can similarly define the notion of <u>cubes/cubing</u>. Like we did last time,

$$2^3 \quad : \quad \text{“ two cube. ”}$$

$$5^3 \quad : \quad \text{“ five cube. ”}$$

$$10^3 \quad : \quad \text{“ ten cube. ”}$$

Now, the meaning of these:

$$2^3 = 2 \cdot 2 \cdot 2 = 8.$$

$$5^3 = 5 \cdot 5 \cdot 5 = 125.$$

$$10^3 = 10 \cdot 10 \cdot 10 = 1000.$$

So cubing of a number means you multiply out three replicas of that number. More generally, x^3 is called “ <u>x cube</u> ”, or sometimes “ <u>the cube of x</u> ”.

$$\boxed{x^3 = x \cdot x \cdot x}.$$

Clearly

$$\boxed{x^3 = x^2 \cdot x}, \text{ or the same } \boxed{x^3 = x \cdot x^2}.$$

Example 0. Find 0^3.

Solution. $$0^3 = 0 \cdot 0 \cdot 0 = 0.$$

Example 1. Find 1^3.

Solution. $$1^3 = 1 \cdot 1 \cdot 1 = 1.$$

Example 2. Find 3^3.

Solution. $$3^3 = 3 \cdot 3 \cdot 3 = 27.$$

Example 3. Find 4^3.

Solution. $$4^3 = 4 \cdot 4 \cdot 4 = 64.$$

Example 4. Find 6^3.

Solution. $$6^3 = 6 \cdot 6 \cdot 6 = 216.$$

Example 5. Find 7^3.

Solution. $$7^3 = 7 \cdot 7 \cdot 7 = 343.$$

Example 6. Find 8^3.

Solution. $$8^3 = 8 \cdot 8 \cdot 8 = 512.$$

Example 7. Find 9^3.

Solution. $$9^3 = 9 \cdot 9 \cdot 9 = 729.$$

Example 8. Find 11^3.

Solution. $$11^3 = 11 \cdot 11 \cdot 11 = 1331.$$

- Cubes of integers between 0—19.

x	0	1	2	3	4	5	6	7	8	9
x^3	0	1	8	27	64	125	216	343	512	729

x	10	11	12	13	14	15	16	17	18	19
x^3	1000	1331	1728	2197	2744	3375	4096	4913	5832	6859

⋆ Get used to expressions like

(a) $\left(x+1\right)^3$,

(b) $\left(x+2\right)^3$,

(c) $\left(x+3\right)^3$,

(d) $\left(x+4\right)^3$,

and so forth.

Example 8. Substitute $\boxed{x = 4}$ in $\boxed{\left(x+1\right)^3}$. Calculate the result.

Solution. $$\left(4+1\right)^3 = 5^3 = 125.$$

Example 9. Substitute $\boxed{x = 5}$ in $\boxed{\left(x+2\right)^3}$. Calculate the result.

Solution. $$\left(5+2\right)^3 = 7^3 = 343.$$

Example 10. Substitute $\boxed{x = 7}$ in $\boxed{\left(x+3\right)^3}$. Calculate the result.

Solution. $$\left(7+3\right)^3 = 10^3 = 1000.$$

- **Expansions.**

(a) $\left(x+\underline{\underline{1}}\right)^3 = x^3 + \underline{\underline{3}}\,x^2 + \underline{\underline{3}}\,x + \underline{\underline{1}}\,.$

(b) $\left(x+\underline{\underline{2}}\right)^3 = x^3 + \underline{\underline{6}}\,x^2 + \underline{\underline{12}}\,x + \underline{\underline{8}}\,.$

(c) $\left(x+\underline{\underline{3}}\right)^3 = x^3 + \underline{\underline{9}}\,x^2 + \underline{\underline{27}}\,x + \underline{\underline{27}}\,.$

(d) $\left(x+\underline{\underline{4}}\right)^3 = x^3 + \underline{\underline{12}}\,x^2 + \underline{\underline{48}}\,x + \underline{\underline{64}}\,.$

(e) $\left(x+\underline{\underline{5}}\right)^3 = x^3 + \underline{\underline{15}}\,x^2 + \underline{\underline{75}}\,x + \underline{\underline{125}}\,.$

(f) $\left(x+\underline{\underline{6}}\right)^3 = x^3 + \underline{\underline{18}}\,x^2 + \underline{\underline{108}}\,x + \underline{\underline{216}}\,.$

(g) $\left(x+\underline{\underline{7}}\right)^3 = x^3 + \underline{\underline{21}}\,x^2 + \underline{\underline{147}}\,x + \underline{\underline{343}}\,.$

(h) $\left(x+\underline{\underline{8}}\right)^3 = x^3 + \underline{\underline{24}}\,x^2 + \underline{\underline{192}}\,x + \underline{\underline{512}}\,.$

(i) $\left(x+\underline{\underline{9}}\right)^3 = x^3 + \underline{\underline{27}}\,x^2 + \underline{\underline{243}}\,x + \underline{\underline{729}}\,.$

(j) $\left(x+\underline{\underline{10}}\right)^3 = x^3 + \underline{\underline{30}}\,x^2 + \underline{\underline{300}}\,x + \underline{\underline{1000}}\,.$

Meaning:

These identities are all true, no matter what number you substitute for x.

Patterns: $\left(x + \boxed{a}\right)^3 = x^3 + \boxed{3\,a}\,x^2 + \boxed{3\,a^2}\,x + \boxed{a^3}$

multiply by 3

square, then multiply by 3

cube

Example 11. Substitute $\boxed{x = 2}$ in each of the two sides of (d). Calculate the results separately. Do they match?

Solution. Let's duplicate (d):

(d) $$\boxed{\left(x+4\right)^3 = x^3 + 12\,x^2 + 48\,x + 64} \,.$$

Once $x = 2$ is substituted,

$$\text{The left-hand side of (d)} = \left(2+4\right)^3 = 6^3 = \boxed{216}\,,$$

$$\text{The right-hand side of (d)} = 2^3 + 12 \cdot 2^2 + 48 \cdot 2 + 64$$

$$= 8 + 48 + 96 + 64 = \boxed{216}\,.$$

• So, as you can see, no matter what number you substitute for x, the formulas (a)–(j) are true. Once again,

Patterns:

$$\boxed{\left(x + a\right)^3 = x^3 + 3\,a\,x^2 + 3\,a^2\,x + a^3} \,.$$

Now, just like last time (when we talked about the square formula), in this cube formula, two letters x and a are involved. Though we originally had x, there arose the necessity to adopt a as the second letter. This is for demonstrating the 'patterns' in the formulas (a)—(j). So, like last time, let's regard the above as a more general formula, one that covers (a)—(j) as ten different special cases. Also, like last time, we could have actually chosen the letter y instead of a. So

Formula A.

$$\boxed{\Big(x + y\Big)^3 \;=\; x^3 \;+\; 3\,x^2\,y \;+\; 3\,x\,y^2 \;+\; y^3} \,.$$

Or, if you prefer a and b, instead of x and y, then that's perfectly fine, the same can be written as

Formula A.

$$\boxed{\Big(a + b\Big)^3 \;=\; a^3 \;+\; 3\,a^2\,b \;+\; 3\,a\,b^2 \;+\; b^3} \,.$$

Once again:

Meaning:

These formulas are all true, no matter what numbers you substitute for x, y, a and b each.

Another related formula:

Formula B.

(a) $$\boxed{x^3 + y^3 \;=\; \Big(x + y\Big)\Big(x^2 - x\,y + y^2\Big)} \,,$$

(b) $$\boxed{x^3 - y^3 \;=\; \Big(x - y\Big)\Big(x^2 + x\,y + y^2\Big)} \,.$$

Example 12. Substitute $\boxed{x = 3}$ and $\boxed{y = 1}$ in each of the two sides of Formula B, part (a). Calculate the results separately. Confrim that they match.

Solution. Let's duplicate Formula B, part (a):

(a) $$\boxed{x^3 + y^3 = \Big(x + y\Big)\Big(x^2 - xy + y^2\Big)}\,.$$

Once $x = 3$ and $y = 1$ are substituted,

$$\text{The left-hand side of (a)} = 3^3 + 1^3 = 27 + 1 = \boxed{28},$$

$$\text{The right-hand side of (a)} = \Big(3+1\Big)\Big(3^2 - 3 \cdot 1 + 1^2\Big)$$

$$= 4 \cdot \Big(9 - 3 + 1\Big) = 4 \cdot 7 = \boxed{28}\,.$$

Example 13. Substitute $\boxed{x = 5}$ and $\boxed{y = 2}$ in each of the two sides of Formula B, part (b). Calculate the results separately. Confrim that they match.

Solution. Let's duplicate Formula B, part (b):

(b) $$\boxed{x^3 - y^3 = \Big(x - y\Big)\Big(x^2 + xy + y^2\Big)}\,.$$

Once $x = 5$ and $y = 2$ are substituted,

$$\text{The left-hand side of (b)} = 5^3 - 2^3 = 125 - 8 = \boxed{117},$$

$$\text{The right-hand side of (b)} = \Big(5-2\Big)\Big(5^2 + 5 \cdot 2 + 2^2\Big)$$

$$= 3 \cdot \Big(25 + 10 + 4\Big) = 3 \cdot 39 = \boxed{117}.$$

⋆ Below are some tweaks (you don't have to memorize them):

Formula C.

$$\left(x + y + z\right)^3 = x^3 + y^3 + z^3 + 3\left(x+y\right)\left(x+z\right)\left(y+z\right).$$

Formula D.

$$x^3 + y^3 + z^3 - 3\,x\,y\,z = \left(x + y + z\right)\left(x^2 + y^2 + z^2 - x\,y - x\,z - y\,z\right).$$

Formula E.

$$x^3 + y^3 + z^3 - \left(x+y\right)\left(x+z\right)\left(y+z\right) + 4\,x\,y\,z = \left(-x+y+z\right)\left(x-y+z\right)\left(x+y-z\right).$$

Formula F.

$$\left(x + y\right)\left(x + z\right)\left(y + z\right) + x\,y\,z = \left(x + y + z\right)\left(x\,y + x\,z + y\,z\right).$$

⋆ Note that Formula F does not involve cubes, but it is related so included.

⋆ In what follows some negative numbers are involved (see Example 14 below).

Formula D′. If a, b and c satisfy

$$\boxed{a + b + c = 0},$$

then

$$\boxed{a^3 + b^3 + c^3 = 3\,a\,b\,c}.$$

Formula D″. If a, b, c, p, q and r satisfy

$$\boxed{\begin{array}{lll} p = a + b, & & \\ q = a + c, & & \text{and} \\ r = b + c & & \end{array}}$$

then

$$\boxed{p^3 + q^3 + r^3 - 3\,p\,q\,r = 2\left(a^3 + b^3 + c^3 - 3\,a\,b\,c\right)}.$$

- **Cubing a negative number.** We have

$$\left(-1\right)^3 = \left(-1\right)\cdot\left(-1\right)\cdot\left(-1\right) = 1\cdot\left(-1\right) = -1,$$
$$\left(-2\right)^3 = \left(-2\right)\cdot\left(-2\right)\cdot\left(-2\right) = 4\cdot\left(-2\right) = -8,$$
$$\left(-3\right)^3 = \left(-3\right)\cdot\left(-3\right)\cdot\left(-3\right) = 9\cdot\left(-3\right) = -27,$$

and so on. More generally,

$$\boxed{\left(-a\right)^3 = -a^3}\ .$$

Example 14. (1) Agree that if you substitute

$$a = 2, \qquad b = 1 \qquad \text{and} \qquad c = -3$$

in $a + b + c$, then the result is 0.

(2) Substitute

$$a = 2, \qquad b = 1 \qquad \text{and} \qquad c = -3$$

in each of $a^3 + b^3 + c^3$, and $3\,a\,b\,c$, and confirm that the results match.

Solution. (1) $2 + 1 + \left(-3\right)$ clearly equals 0.

(2) $a^3 + b^3 + c^3 = 2^3 + 1^3 + \left(-3\right)^3 = 8 + 1 + \left(-27\right) = -18,$

whereas $3\,a\,b\,c = 3\cdot 2\cdot 1\cdot\left(-3\right) = -18.$

Example 15. (1) Substitute

$$a = 3, \qquad b = -2 \qquad \text{and} \qquad c = 4$$

in

$$\boxed{\begin{aligned} p &= a + b, \\ q &= a + c, \qquad \text{and} \\ r &= b + c \end{aligned}}$$

(2) Substitute the values a, b, c, p, q and r as in (1) in each of

$$\boxed{p^3 + q^3 + r^3 - 3\,p\,q\,r} \qquad \text{and} \qquad \boxed{2\,\left(a^3 + b^3 + c^3 - 3\,a\,b\,c\right)},$$

and confirm that the results match.

Solution. (1) $p = 3 + \left(-2\right) = 1$. $q = 3 + 4 = 7$. $r = \left(-2\right) + 4 = 2$.

(2) Substituting $p = 1$, $q = 7$ and $r = 2$ in

$$\boxed{p^3 + q^3 + r^3 - 3\,p\,q\,r}$$

yields

$$1^3 + 7^3 + 2^3 - 3 \cdot 1 \cdot 7 \cdot 2 = 1 + 343 + 8 - 42 = 310.$$

Meanwhile, substituting $a = 3$, $b = -2$ and $c = 4$ in

$$\boxed{2\,\left(a^3 + b^3 + c^3 - 3\,a\,b\,c\right)}$$

yields

$$\begin{aligned} 2 \cdot \left(3^3 + \left(-2\right)^3 + 4^3 - 3 \cdot 3 \cdot \left(-2\right) \cdot 4\right) &= 2 \cdot \left(27 - 8 + 64 + 72\right) \\ &= 2 \cdot 155 = 310. \end{aligned}$$

$\star$ Just like the last item in the previous set of notes ("Review of Lectures – V"), you don't have to know the following (yet), as it involves a determinant. Some of you might have seen it before thus included. We might cover determinants later.

Formula G.*

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a^3 + b^3 + c^3 - 3\,a\,b\,c.$$

Formula H.

$$\begin{vmatrix} a+b+c & 3\,a & 3\,a \\ 3\,b & a+b+c & 3\,a \\ 3\,b & 3\,b & a+b+c \end{vmatrix} + 27\,a\,b\,c = \left(a+b+c\right)^3.$$

Formula I.

$$\begin{vmatrix} a^2+b^2-c^2-d^2 & 2\bigl(bc+ad\bigr) & 2\bigl(bd-ac\bigr) \\ 2\bigl(bc-ad\bigr) & a^2-b^2+c^2-d^2 & 2\bigl(cd+ab\bigr) \\ 2\bigl(bd+ac\bigr) & 2\bigl(cd-ab\bigr) & a^2-b^2-c^2+d^2 \end{vmatrix}$$

$$= \left(a^2+b^2+c^2+d^2\right)^3.$$

*Suggested by my colleague, Professor Pavlos Tzermias.