Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – V

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§5. Squares.

• Today we learn squares/squaring . This is quintessentially important. I cannot imagine math without squares. First, notation and pronunciation:

3^2 : "three square."

5^2 : "five square."

10^2 : "ten square."

Now, the meaning of these:

$$3^2 = 3 \cdot 3 = 9.$$

$$5^2 = 5 \cdot 5 = 25.$$

$$10^2 = 10 \cdot 10 = 100.$$

So squaring of a number means you multiply out two replicas of that number. More generally, x^2 is called " x square ", or sometimes " the square of x ". It means you multiply x with x. So

$$\boxed{x^2 = x \cdot x}.$$

Question. Why not just write $x \cdot x$?

The notation x^2 is more convenient. We will appreciate this notation more as the semester progresses.

Example 0. Find 0^2.

Solution.
$$0^2 = 0 \cdot 0 = 0.$$

Example 1. Find 1^2.

Solution.
$$1^2 = 1 \cdot 1 = 1.$$

Example 2. Find 2^2.

Solution.
$$2^2 = 2 \cdot 2 = 4.$$

Example 3. Find 4^2.

Solution.
$$4^2 = 4 \cdot 4 = 16.$$

Example 4. Find 6^2.

Solution.
$$6^2 = 6 \cdot 6 = 36.$$

Example 5. Find 7^2.

Solution. $$7^2 = 7 \cdot 7 = 49.$$

Example 6. Find 8^2.

Solution. $$8^2 = 8 \cdot 8 = 64.$$

Example 7. Find 9^2.

Solution. $$9^2 = 9 \cdot 9 = 81.$$

- Squares of integers between 0—19.

x	0	1	2	3	4	5	6	7	8	9
x^2	0	1	4	9	16	25	36	49	64	81

x	10	11	12	13	14	15	16	17	18	19
x^2	100	121	144	169	196	225	256	289	324	361

⋆ Get used to expressions like

(a) $\left(x+1\right)^2$,

(b) $\left(x+2\right)^2$,

(c) $\left(x+3\right)^2$,

(d) $\left(x+4\right)^2$,

and so forth.

Example 8. Substitute $\boxed{x = 3}$ in $\boxed{\left(x+1\right)^2}$. Calculate the result.

Solution. $$\left(3+1\right)^2 = 4^2 = 16.$$

Example 9. Substitute $\boxed{x = 6}$ in $\boxed{\left(x+2\right)^2}$. Calculate the result.

Solution. $$\left(6+2\right)^2 = 8^2 = 64.$$

Example 10. Substitute $\boxed{x = 8}$ in $\boxed{\left(x+3\right)^2}$. Calculate the result.

Solution. $$\left(8+3\right)^2 = 11^2 = 121.$$

- **Expansions.**

(a) $\left(x+\underline{\underline{1}}\right)^2 = x^2 + \underline{\underline{2}}\,x + \underline{\underline{1}}$.

(b) $\left(x+\underline{\underline{2}}\right)^2 = x^2 + \underline{\underline{4}}\,x + \underline{\underline{4}}$.

(c) $\left(x+\underline{\underline{3}}\right)^2 = x^2 + \underline{\underline{6}}\,x + \underline{\underline{9}}$.

(d) $\left(x+\underline{\underline{4}}\right)^2 = x^2 + \underline{\underline{8}}\,x + \underline{\underline{16}}$.

(e) $\left(x+\underline{\underline{5}}\right)^2 = x^2 + \underline{\underline{10}}\,x + \underline{\underline{25}}$.

(f) $\left(x+\underline{\underline{6}}\right)^2 = x^2 + \underline{\underline{12}}\,x + \underline{\underline{36}}$.

(g) $\left(x+\underline{\underline{7}}\right)^2 = x^2 + \underline{\underline{14}}\,x + \underline{\underline{49}}$.

(h) $\left(x+\underline{\underline{8}}\right)^2 = x^2 + \underline{\underline{16}}\,x + \underline{\underline{64}}$.

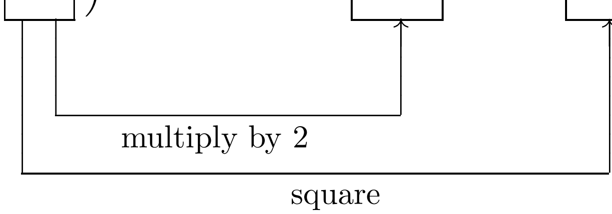
(i) $\left(x+\underline{\underline{9}}\right)^2 = x^2 + \underline{\underline{18}}\,x + \underline{\underline{81}}$.

(j) $\left(x+\underline{\underline{10}}\right)^2 = x^2 + \underline{\underline{20}}\,x + \underline{\underline{100}}$.

Meaning:

$\underline{\underline{\text{These identities are all true, no matter what number you substitute for } x.}}$

Patterns: $\left(x + \boxed{a}\right)^2 = x^2 + \boxed{2\,a}\;x + \boxed{a^2}$



Example 11. Substitute $\boxed{x = 2}$ in each of the two sides of (e). Calculate the results separately. Do they match?

Solution. Let's duplicate (e):

(e) $$\boxed{\left(x+5\right)^2 = x^2 + 10\,x + 25}\,.$$

Once $x = 2$ is substituted,

$$\text{The left-hand side of (e)} = \left(2+5\right)^2 = 7^2 = \boxed{49}\,,$$

$$\text{The right-hand side of (e)} = 2^2 + 10 \cdot 2 + 25 = 4 + 20 + 25 = \boxed{49}\,.$$

Example 12. Substitute $\boxed{x = 3}$ in each of the two sides of (g). Calculate the results separately. Do they match?

Solution. Let's duplicate (g):

(g) $$\boxed{\left(x+7\right)^2 = x^2 + 14\,x + 49}\,.$$

Once $x = 3$ is substituted,

$$\text{The left-hand side of (g)} = \left(3+7\right)^2 = 10^2 = \boxed{100}\,,$$

$$\text{The right-hand side of (g)} = 3^2 + 14 \cdot 3 + 49 = 9 + 42 + 49 = \boxed{100}\,.$$

• So, as you can see, no matter what number you substitute for x, the formulas (a)–(j) are true. Once again,

Patterns:

$$\boxed{\left(x + a\right)^2 = x^2 + 2\,a\,x + a^2} \,.$$

Now, in this formula two letters x and a are involved. Actually, x was the original letter which we were using. Then there arose the necessity to adopt another letter a for the purpose of demonstrating the 'patterns' in the formulas (a)—(j) (in page 5). Now, we may regard the above as a more general formula, a formula that covers (a)—(j) as ten different special cases at once. In other words, we may substitute any number you like for x and for a each in the above formula. Here, there is no logical reason we should choose a as the second letter. What's next to x in the alphabet is y, so we could have actually written the formula as

Formula A.

$$\boxed{\left(x + y\right)^2 = x^2 + 2\,x\,y + y^2} \,.$$

Or, if you prefer a and b, instead of x and y, then that's perfectly fine, the same can be written as

Formula A.

$$\boxed{\left(a + b\right)^2 = a^2 + 2\,a\,b + b^2} \,.$$

Once again:

Meaning:

These formulas are all true, no matter what numbers you substitute for x, y, a and b each.

In what follows, I sometimes use $x, y, z, ..$, and some other times $a, b, c, ..$

⋆ Now, here is another related formula (very important):

Formula B.

$$\boxed{x^2 - y^2 = \left(x + y\right)\left(x - y\right)}\ .$$

⋆ Here are some tweaks:

Formula C.

$$\boxed{\left(x + y\right)^2 - \left(x - y\right)^2 = 4\,x\,y}\ .$$

Formula D.

$$\boxed{\left(x^2 + y^2\right)^2 - 4\,x^2\,y^2 = \left(x + y\right)^2\left(x - y\right)^2}\ .$$

Formula E.

$$\left(x^2 + y^2 + z^2\right)^2 - 4\left(x^2\,y^2 + x^2\,z^2 + y^2\,z^2\right)$$
$$= \left(x+y+z\right)\cdot\left(x-y-z\right)\cdot\left(-x+y-z\right)\cdot\left(-x-y+z\right).$$

⋆ Formula E has a connection with geometry. More precisely, the two sides of Formula E are precisely the negative of the square of the area of the triangle whose edges have lengths x, y and z (Heron's formula).

⋆ The following is a tweak of Formula E.

Formula F.

$$\left(a^2 - b^2 - c^2 + d^2 \right)^2 - 4 \left(a\,d - b\,c \right)^2$$

$$= \left(a^2 + b^2 + c^2 + d^2 \right)^2$$

$$- 4 \left(a^2\,b^2 + a^2\,c^2 + a^2\,d^2 + b^2\,c^2 + b^2\,d^2 + c^2\,d^2 \right)$$

$$+ 8 \; a \; b \; c \; d$$

$$= \left(a + b + c + d \right) \cdot \left(a + b - c - d \right)$$

$$\cdot \left(a - b + c - d \right) \cdot \left(a - b - c + d \right)$$

$$\left(= \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} \right).$$

⋆ In the last one, you see a formation (enclosed in the big parenthesis)

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} \quad .$$

You don't have to know what it is (yet). It is called <u>the determinant</u>. Some of you might have seen it somewhere. We might go over the basics of determinants later in the semester (if the time permits).