Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – IV

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§4. Pascal's Triangle.

Let's summarize our discussion so far: We started with

$$
\begin{aligned}
&1 = 1,\\
&1 + 2 = 3,\\
&1 + 2 + 3 = 6,\\
&1 + 2 + 3 + 4 = 10,\\
&1 + 2 + 3 + 4 + 5 = 15,\\
&1 + 2 + 3 + 4 + 5 + 6 = 21,\\
&1 + 2 + 3 + 4 + 5 + 6 + 7 = 28,\\
&1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36,\\
&1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45,\\
&\qquad\qquad \vdots \qquad\qquad\qquad\qquad \ddots\\
&1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + \cdots + n \quad = \frac{1}{2}\,n\Big(n+1\Big)
\end{aligned}
$$

$\Big($where the last line is the formula we pulled$\Big)$. To align the answers:

$$1, \quad 3, \quad 6, \quad 10, \quad 15, \quad 21, \quad 28, \quad 36, \quad 45, \ \cdots \ , \ \frac{1}{2}\,n\Big(n+1\Big).$$

We added these up:

$$\begin{aligned}
1 &= 1,\\
1+3 &= 4,\\
1+3+6 &= 10,\\
1+3+6+10 &= 20,\\
1+3+6+10+15 &= 35,\\
1+3+6+10+15+21 &= 56,\\
1+3+6+10+15+21+28 &= 84,\\
1+3+6+10+15+21+28+36 &= 120,\\
\vdots \qquad & \qquad \ddots\\
1+3+6+10+\cdots+\frac{1}{2}n\Big(n+1\Big) &= \frac{1}{6}\,n\Big(n+1\Big)\Big(n+2\Big).
\end{aligned}$$

$\Big($where, again, the last line is the formula we pulled$\Big)$. To align the answers:

$$1, \quad 4, \quad 10, \quad 20, \quad 35, \quad 56, \quad 84, \quad 120, \ \cdots, \quad \frac{1}{6}\,n\Big(n+1\Big)\Big(n+2\Big).$$

Now, have you thought about what if we continue the process, as in adding these up:

$$\begin{array}{ll}
(1) & 1 \ =? \\
(2) & 1+4 \ =? \\
(3) & 1+4+10 \ =? \\
(4) & 1+4+10+20 \ =? \\
(5) & 1+4+10+20+35 \ =? \\
(6) & 1+4+10+20+35+56 \ =? \\
(7) & 1+4+10+20+35+56+84 \ =? \\
(8) & 1+4+10+20+35+56+84+120 \ =? \\
(9) & 1+4+10+20+35+56+84+120+165 \ =? \\
\vdots & \qquad \vdots \qquad\qquad\qquad\qquad \ddots \\
(n) & 1+4+10+20+35+56+84+120+\cdots+\frac{1}{6}n\Big(n+1\Big)\Big(n+2\Big) \ =?
\end{array}$$

Have you wondered about the shape of the formula for the line (n)?

Now, you would say

> "*Hey, the first few can be done brute-force, but you are probably going to show us a trick how to do it for like part* (100), *and that is probably more complicated than the last one, and following all that is not very enticing..*"

Yeah, you are right. That's what I had in mind doing next. Actually it is not as bad as you think. So, bear with me. Because it has some bearings on our later material. I will make it quick. I will do it with an arbitrary n. Don't get too overwhelmed, or obsessed with the idea to understand every inch of it, until the end of page 6.

[Clue] We are supposed to derive a formula for

$$1 + 4 + 10 + 20 + 35 + 56 + \cdots + \frac{1}{6}n\Big(n+1\Big)\Big(n+2\Big).$$

For that matter, keep in mind

$$\begin{array}{rcl}
1 & = & 1 \\
4 & = & 1+3 \\
10 & = & 1+3+6 \\
20 & = & 1+3+6+10 \\
35 & = & 1+3+6+10+15 \\
56 & = & 1+3+6+10+15+21 \\
\vdots & & \vdots \qquad\qquad\qquad\qquad \ddots \\
\frac{1}{6}n(n+1)(n+2) & = & 1+3+6+10+15+\cdots+\frac{1}{2}n(n+1) \\
\hline
\end{array}$$

+)

$$1+4+10+20+35+\cdots+\frac{1}{6}n(n+1)(n+2) = \begin{array}{l}
\ \ 1 \\
+1+3 \\
+1+3+6 \\
+1+3+6+10 \\
+1+3+6+10+15 \\
+1+3+6+10+15+21 \\
\ \ \vdots \qquad\qquad\qquad\qquad \ddots \\
+1+3+6+10+15+\cdots+\frac{1}{2}n(n+1)
\end{array}$$

So we end up calculating

$$\boxed{\begin{array}{l} \quad 1 \\ +1+3 \\ +1+3+6 \\ +1+3+6+10 \\ +1+3+6+10+15 \\ \quad\vdots \qquad\qquad\qquad\qquad \ddots \\ +1+3+6+10+15+\cdots+\frac{1}{2}n(n+1) \end{array}} \quad \longleftarrow \quad \underline{\underline{\text{Let's call it}}} \quad \underline{\underline{\boxed{x}}}$$

though it is still helpful to remember

$$\boxed{x} \;=\; \boxed{1 + 4 + 10 + 20 + 35 + \cdots + \frac{1}{6}\, n\left(n+1\right)\left(n+2\right)}\,.$$

[**Solution**] x equals

$$= \boxed{\begin{array}{llll} \quad 1 & +3+6+10+15+ & \cdots & +\frac{1}{2}(n+1)(n+2) \\ +1+3 & \quad +6+10+15+ & \cdots & +\frac{1}{2}(n+1)(n+2) \\ +1+3+6 & \quad +10+15+ & \cdots & +\frac{1}{2}(n+1)(n+2) \\ +1+3+6+10 & \quad +15+ & \cdots & +\frac{1}{2}(n+1)(n+2) \\ +1+3+6+10+15 & \quad + & \cdots & +\frac{1}{2}(n+1)(n+2) \\ \quad\vdots & & \ddots & \quad\vdots \\ +1+3+6+10+ & \cdots+\frac{1}{2}n(n+1) & & +\frac{1}{2}(n+1)(n+2) \end{array}} \quad \underline{\text{Box } \#1}$$

$$- \boxed{\begin{array}{r} 3+6+10+15+\cdots+\frac{1}{2}(n+1)(n+2) \\ +6+10+15+\cdots+\frac{1}{2}(n+1)(n+2) \\ +10+15+\cdots+\frac{1}{2}(n+1)(n+2) \\ +15+\cdots+\frac{1}{2}(n+1)(n+2) \\ \ddots \qquad \vdots \qquad \\ +\frac{1}{2}(n+1)(n+2) \end{array}} \quad \underline{\text{Box } \#2}$$

First, Box #1 is calculated simply as

$$\left.\begin{array}{l}
\boxed{\quad 1 + 3 + 6 + 10 + 15 + \cdots + \tfrac{1}{2}(n{+}1)(n{+}2)} \\
\boxed{+\ 1 + 3 + 6 + 10 + 15 + \cdots + \tfrac{1}{2}(n{+}1)(n{+}2)} \\
\boxed{+\ 1 + 3 + 6 + 10 + 15 + \cdots + \tfrac{1}{2}(n{+}1)(n{+}2)} \\
\boxed{+\ 1 + 3 + 6 + 10 + 15 + \cdots + \tfrac{1}{2}(n{+}1)(n{+}2)} \\
\qquad\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
\boxed{+\ 1 + 3 + 6 + 10 + 15 + \cdots + \tfrac{1}{2}(n{+}1)(n{+}2)}
\end{array}\right]
\begin{array}{c}
n \\ \text{identical} \\ \text{boxes,} \\ \text{each} \\ \text{equals} \\ \tfrac{1}{6}(n{+}1)(n{+}2)(n{+}3) \\ \uparrow \\ \boxed{\begin{array}{c}\text{Shift } \ n \mapsto n{+}1 \ \text{ in} \\ \tfrac{1}{6}n(n{+}1)(n{+}2)\end{array}}
\end{array}$$

So

$$\begin{aligned}
\underline{\text{Box }\#1} &= n \cdot \frac{1}{6}\Big(n+1\Big)\Big(n+2\Big)\Big(n+3\Big) \\
&= \frac{1}{6}\,n\Big(n+1\Big)\Big(n+2\Big)\Big(n+3\Big)\,.
\end{aligned}$$

Meanwhile, Box #2 equals

$$\begin{array}{ccccccccc}
\boxed{3} & \boxed{\begin{array}{c}+\ 6\\+\ 6\end{array}} & \boxed{\begin{array}{c}+\ 10\\+\ 10\\+\ 10\end{array}} & \boxed{\begin{array}{c}+\ 15\\+\ 15\\+\ 15\\+\ 15\end{array}} & \boxed{\begin{array}{c}+\ 21\\+\ 21\\+\ 21\\+\ 21\\+\ 21\end{array}} & \begin{array}{c}+\ \cdots\ +\\+\ \cdots\ +\\+\ \cdots\ +\\+\ \cdots\ +\\+\ \cdots\ +\\ \ddots\end{array} & \boxed{\begin{array}{c}\tfrac{1}{2}(n{+}1)(n{+}2)\\\tfrac{1}{2}(n{+}1)(n{+}2)\\\tfrac{1}{2}(n{+}1)(n{+}2)\\\tfrac{1}{2}(n{+}1)(n{+}2)\\\tfrac{1}{2}(n{+}1)(n{+}2)\\\vdots\\\tfrac{1}{2}(n{+}1)(n{+}2)\end{array}} & \longleftarrow & \begin{array}{c}\text{there}\\\text{are}\\n\\\text{of}\\\tfrac{1}{2}(n{+}1)(n{+}2)\end{array} \\
\hline
+) & & & & & & & & \\
3 & +\ 12 & +\ 30 & +\ 60 & +\ 105 & +\ \cdots\ + & \tfrac{1}{2}n(n{+}1)(n{+}2) & & \\
\| & \| & \| & \| & \| & & \| & & \\
3\cdot\underline{\underline{1}} & 3\cdot\underline{\underline{4}} & 3\cdot\underline{\underline{10}} & 3\cdot\underline{\underline{20}} & 3\cdot\underline{\underline{35}} & & 3\cdot\underline{\underline{\tfrac{1}{6}n(n{+}1)(n{+}2)}} & &
\end{array}$$

In other words, Box #2 equals

$$3 \quad \Big(\underbrace{1 \ + \ 4 \ + \ 10 \ + \ 20 \ + \cdots + \ \tfrac{1}{6}n(n+1)(n+2)}_{\textstyle \overset{\|}{x}} \Big)$$

Thus

$$\underline{\text{Box } \#2} \quad = \quad 3\,x.$$

Remember,

$$x \quad = \quad \underbrace{\underline{\text{Box } \#1}}_{\textstyle \overset{\|}{\tfrac{1}{6}n(n+1)(n+2)(n+3)}} \quad - \quad \underbrace{\underline{\text{Box } \#2}}_{\textstyle \overset{\|}{3\,x}}$$

So this reads

$$x \quad = \quad \frac{1}{6}\,n\,\Big(n+1\Big)\,\Big(n+2\Big)\,\Big(n+3\Big) \quad - \quad 3\,x \qquad \Big(\underline{\text{key equation}}\Big)$$

This is readily solved as

$$x \ = \ \frac{1}{24}\,n\,\Big(n+1\Big)\,\Big(n+2\Big)\,\Big(n+3\Big).$$

[End of solution].

And that's it. So, between page 3 and here, you will not be asked to duplicate it and explain it. The reason I did it despite that, and also the fact that this is not deemed 'appealing' any more, as in this was essentially a repeat of our last class with some necessary tweak, is because I wanted to see the patterns. Now, our perseverance paid off. We got hold of the patterns in how the key equation changes, and thus the solution x. I will elaborate this after highlighting the formula which we just got hold of. Here is the formula (this one I expect you to fully understand):

Formula. Let n be a positive integer. Then

$$\boxed{\begin{aligned} 1 + \underbrace{4}_{\substack{\| \\ 1+3}} + \underbrace{10}_{\substack{\| \\ 1+3+6}} + \underbrace{20}_{\substack{\| \\ 1+3+6+10}} + \cdots + \underbrace{\frac{1}{6}n\Big(n+1\Big)\Big(n+2\Big)}_{\substack{\| \\ 1+3+6+10+\cdots+\frac{1}{2}(n+1)(n+2)}} \\ = \quad \frac{1}{24}\, n\left(n+1\right)\left(n+2\right)\left(n+3\right). \end{aligned}}$$

Formula paraphrased. Let n be a positive integer. Then

$$\boxed{\begin{array}{llllll} 1 & +1 & +1 & +1 & & +1 \\ & +1+2 & +1+2 & +1+2 & & +1+2 \\ & & +1+2+3 & +1+2+3 & \cdots & +1+2+3 \\ & & & +1+2+3+4 & & +1+2+3+4 \\ & & & & & \vdots \qquad \ddots \\ & & & & & +1+2+3+4+\cdots+n \\ & & & & & \\ & & & = & \multicolumn{2}{l}{\frac{1}{24}\, n\left(n+1\right)\left(n+2\right)\left(n+3\right).} \end{array}}$$

So, where are we going? Like I said, patterns are slowly emerging. It is good to dissect the denominator 24. This denominator came out as a result of solving

$$x \quad = \quad \frac{1}{6}\, n\left(n+1\right)\left(n+2\right)\left(n+3\right) \quad - \quad 3\,x \qquad \left(\underline{\text{key equation}}\right).$$

Namely, after dragging $3\,x$ to the other side of the '$=$', the equation becomes

$$4\,x \quad = \quad \frac{1}{6}\, n\left(n+1\right)\left(n+2\right)\left(n+3\right).$$

So, one fourth of $\dfrac{1}{6}$ is $\dfrac{1}{24}$. That's how 24 showed up.

So, we got hold of

$$\begin{array}{ll}
(1) & 1 \;=\; 1, \\
(2) & 1 + 4 \;=\; 5, \\
(3) & 1 + 4 + 10 \;=\; 15, \\
(4) & 1 + 4 + 10 + 20 \;=\; 35, \\
(5) & 1 + 4 + 10 + 20 + 35 \;=\; 70, \\
(6) & 1 + 4 + 10 + 20 + 35 + 56 \;=\; 126, \\
(7) & 1 + 4 + 10 + 20 + 35 + 56 + 84 \;=\; 210, \\
(8) & 1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 \;=\; 330, \\
(9) & 1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 \;=\; 495, \\
\vdots & \qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \ddots \\
(n) & 1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + \cdots + \frac{1}{6} n \Big(n+1\Big) \Big(n+2\Big) \\
& \qquad\qquad\qquad\qquad = \; \frac{1}{24}\, n \, \Big(n+1\Big) \, \Big(n+2\Big) \, \Big(n+3\Big).
\end{array}$$

⋆ Now, let's align these answers

$$1, \quad 5, \quad 15, \quad 35, \quad 70, \quad 126, \quad 210, \quad 330, \quad \cdots \;, \quad \frac{1}{24}\, n \Big(n+1\Big) \Big(n+2\Big) \Big(n+3\Big).$$

Then why don't you add these up, as in

$$1 + 5 + 15 + 35 + 70 + 126 + 210 + 330 + \cdots + \frac{1}{24} n \Big(n+1\Big) \Big(n+2\Big) \Big(n+3\Big),$$

and try to figure out the formula for it? Employ the exact same method as above, with some obvious tweak. That works. And if you actually care to do that, then you will end up solving the new equation

$$x \quad = \quad \frac{1}{24}\, n \, \Big(n+1\Big) \, \Big(n+2\Big) \, \Big(n+3\Big) \, \Big(n+4\Big) \quad - \quad 4\, x.$$

After dragging $4\,x$ to the other side of the '=', the equation will become

$$5\,x \quad = \quad \frac{1}{24}\, n \, \Big(n+1\Big) \, \Big(n+2\Big) \, \Big(n+3\Big) \, \Big(n+4\Big).$$

This way, one fifth of $\dfrac{1}{24}$ which is $\dfrac{1}{120}$, will show up as a part of the answer.

The whole answer will actually be

$$\frac{1}{120} n \Big(n+1\Big) \Big(n+2\Big) \Big(n+3\Big) \Big(n+4\Big).$$

And, like I said, 120 arose as 5 times 24. But 24 arose as 4 times 6. Furthermore, 6 arose as 3 times 2. So, in short,

$$\begin{aligned} 120 &= 5 \cdot 24, \\ 24 &= 4 \cdot 6, \\ 6 &= 3 \cdot 2, \\ 2 &= 2 \cdot 1. \end{aligned}$$

If you care to combine these four lines, then it transpires

$$\begin{aligned} 120 &= 5 \cdot 24 \\ &= 5 \cdot 4 \cdot 6 \\ &= 5 \cdot 4 \cdot 3 \cdot 2 \\ &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1. \end{aligned}$$

So we could have written the above answer as

$$\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} n \Big(n+1\Big) \Big(n+2\Big) \Big(n+3\Big) \Big(n+4\Big),$$

or, the same to say,

$$\frac{n \Big(n+1\Big) \Big(n+2\Big) \Big(n+3\Big) \Big(n+4\Big)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}.$$

And we have come just far enough to say we are achieving something — we are finally unraveling the basic patterns of sums of sequence of integers that naturally arise from $1,\ 2,\ 3,\ 4,\ 5,\ \cdots$

Namely, we have the complete breakdown of the process

$$\begin{array}{cccccccccccc}
 & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & \cdots & \text{(row 1)} \\
\longrightarrow & 1, & 3, & 6, & 10, & 15, & 21, & 28, & 36, & 45, & \cdots & \text{(row 2)} \\
\longrightarrow & 1, & 4, & 10, & 20, & 35, & 56, & 84, & 120, & 165, & \cdots & \text{(row 3)} \\
\longrightarrow & 1, & 5, & 15, & 35, & 70, & 126, & 210, & 330, & 495, & \cdots & \text{(row 4)} \\
\longrightarrow & 1, & 6, & 21, & 56, & 126, & 252, & 462, & 792, & 1287, & \cdots & \text{(row 5)} \\
\longrightarrow & 1, & 7, & 28, & 84, & 210, & 462, & 924, & 1716, & 3003, & \cdots & \text{(row 6)} \\
\longrightarrow & 1, & 8, & 36, & 120, & 330, & 792, & 1716, & 3432, & 6435, & \cdots & \text{(row 7)} \\
\longrightarrow & 1, & 9, & 45, & 165, & 495, & 1287, & 3003, & 6435, & 12870, & \cdots & \text{(row 8)} \\
\longrightarrow & \cdots & & & & & & & & & &
\end{array}$$

By the way, do you notice the same numbers show up at different spots? More on that later. We now have the formula for the numbers that show up in this table. Namely:

- (row 1) from left to right:

  substitute $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots$ in $\dfrac{n}{1} \quad \Big(= n \Big).$

- (row 2) from left to right:

  substitute $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots$ in $\dfrac{n\,\big(n+1\big)}{1 \cdot 2}.$

- (row 3) from left to right:

  substitute $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots$ in $\dfrac{n\,\big(n+1\big)\big(n+2\big)}{1 \cdot 2 \cdot 3}.$

- (row 4) from left to right:

  substitute $n = 1,2,3,4,5,6,7,8,9,\cdots$ in $\dfrac{n\left(n+1\right)\left(n+2\right)\left(n+3\right)}{1\cdot 2\cdot 3\cdot 4}$.

- (row 5) from left to right:

  substitute $n = 1,2,3,4,5,6,7,8,9,\cdots$ in

$$\frac{n\left(n+1\right)\left(n+2\right)\left(n+3\right)\left(n+4\right)}{1\cdot 2\cdot 3\cdot 4\cdot 5}.$$

- (row 6) from left to right:

  substitute $n = 1,2,3,4,5,6,7,8,9,\cdots$ in

$$\frac{n\left(n+1\right)\left(n+2\right)\left(n+3\right)\left(n+4\right)\left(n+5\right)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}.$$

- (row 7) from left to right:

  substitute $n = 1,2,3,4,5,6,7,8,9,\cdots$ in

$$\frac{n\left(n+1\right)\left(n+2\right)\left(n+3\right)\left(n+4\right)\left(n+5\right)\left(n+6\right)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}.$$

- (row 8) from left to right:

  substitute $n = 1,2,3,4,5,6,7,8,9,\cdots$ in

$$\frac{n\left(n+1\right)\left(n+2\right)\left(n+3\right)\left(n+4\right)\left(n+5\right)\left(n+6\right)\left(n+7\right)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}.$$

Actually, notice that, at every spot in the above chart, that number is precisely the number right above it plus the number next to the left. Taking that into account, we are better off rearranging the chart as

1 (row 0)
1 1 (row 1)
1 2 1 (row 2)
1 3 3 1 (row 3)
1 4 6 4 1 (row 4)
1 5 10 10 5 1 (row 5)
1 6 15 20 15 6 1 (row 6)
1 7 21 35 35 21 7 1 (row 7)

I stress that this is a mere rearrangement: The rows in the old chart (page 10) are now parallel lateral lines (as indicated by slashes) in this new, triangle-shaped, chart. The only thing essentially new here is I actually inserted an extra (lateral) line 1–1–1–1–1– $\cdots$ on top.

Caution. I had to re-assign the row numbers. In what follows the row numbers that I refer to are the ones as indicated in the above triangle-shaped chart, (not one on page 10).

Now, needless to say, there is the next horizontal row that is not shown. Once that row is constructed, it has the next row, and so on, it continues endlessly. So this pyramid has infinitely many layers. Now, in this new pyramid, realize that, you pick any spot, and that number equals the sum of two numbers right above it. So it makes sense to re-draw it like:

1								(row 0)
1	1							(row 1)
1	2	1						(row 2)
1	3	3	1					(row 3)
1	4	6	4	1				(row 4)
1	5	10	10	5	1			(row 5)
1	6	15	20	15	6	1		(row 6)
1	7	21	35	35	21	7	1	(row 7)

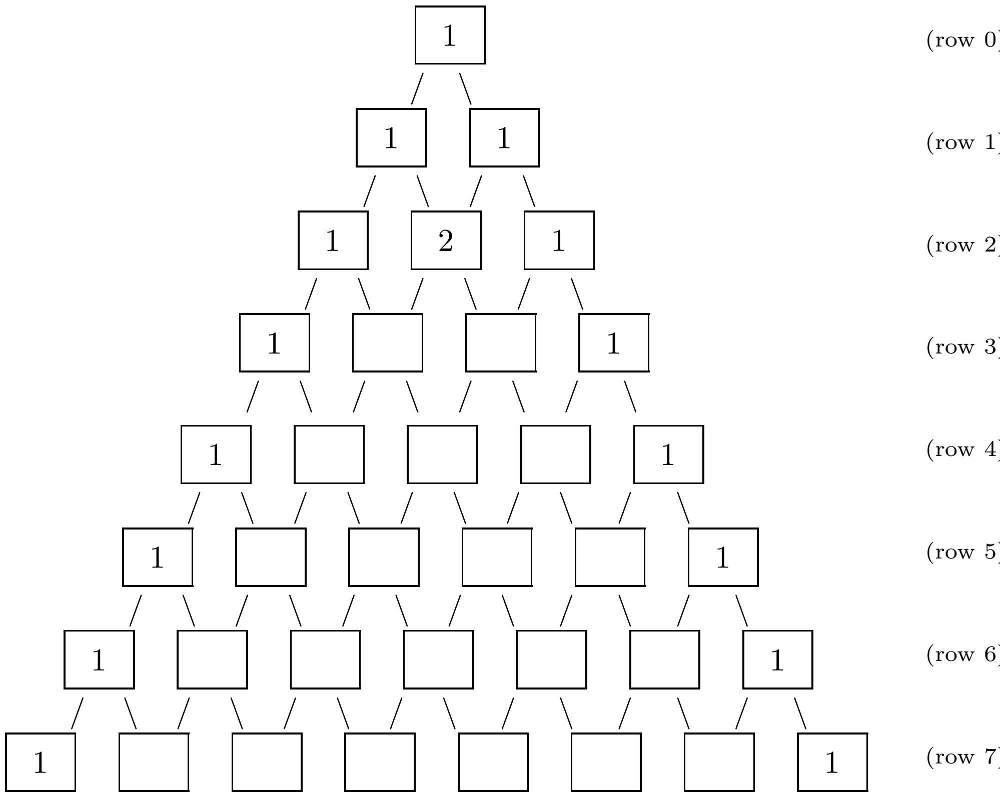
So, once again:

Rule.

At every spot, that number equals the sum of two numbers right above it.

Now, observe the symmetry. This pyramid possesses the left-and-right symmetry. And that's apparently thanks to the aforementioned rule. This explains why the same numbers show up twice at different spots (which was not easily seen from the previous chart). Now, the same rule actually allows us to algorithmically generate the next row, and so on and so forth. But then, suppose you need to know the numbers occupied in the 10-th row, 25-th row, 50-th row, 100-th row, and it takes a lot of time to manually do that. But guess what, you already know that there is a way to go around it. Rely on the formula. By the way, the above pyramid is so famous and so fundamental in math that it has a name. It is called the Pascal's triangle.

Exercise 1. Recover the emptied Pascal's triangle below:



Exercise 2.

(a) Find the third spot from the left in row 9 in the Pascal's triangle.

(b) Find the fourth spot from the left in row 10 in the Pascal's triangle.

Exercise 3.

(a) Write out the eighth spot from the left in row 20 in the Pascal's triangle, in a fraction form. You don't have to simplify the fraction.

(b) Write out the twelfth spot from the left in row 50 in the Pascal's triangle, in a fraction form. You don't have to simplify the fraction.

Answers for Exercise 2 (a) 36. (b) 120.

$\star$ These answers can also be found as follows:

(a) $\dfrac{8 \cdot 9}{1 \cdot 2} = 36.$ (b) $\dfrac{8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3} = 120.$

Answers for Exercise 3

(a) $$\frac{14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}.$$

(b) $$\frac{40 \cdot 41 \cdot 42 \cdot 43 \cdot 44 \cdot 45 \cdot 46 \cdot 47 \cdot 48 \cdot 49 \cdot 50}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11}.$$