

**Math 105 TOPICS IN MATHEMATICS  
REVIEW OF LECTURES – XXXVII**

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§37. TRIGONOMETRY – VI.

Today's first topic is

“congruence.”

We say

- 5 is congruent to 1 modulo 4, because  $5 - 1 = 4$  is divisible by 4.
- 8 is congruent to 1 modulo 7, because  $8 - 1 = 7$  is divisible by 7.
- 29 is congruent to 5 modulo 8, because  $29 - 5 = 24$  is divisible by 8.
- 32 is congruent to 2 modulo 15, because  $32 - 2 = 30$  is divisible by 15.
- 15 is congruent to 3 modulo 6, because  $15 - 3 = 12$  is divisible by 6.
- 28 is congruent to 0 modulo 7, because  $28 - 0 = 28$  is divisible by 7.

Write these as

$$\begin{array}{lll} 5 \equiv 1, & 8 \equiv 1, & 29 \equiv 5, \\ 32 \equiv 2, & 15 \equiv 3, & 28 \equiv 0. \end{array}$$

- More generally, let  $a$ ,  $b$  and  $r$  be integers, and  $r \geq 2$ . We say

“ $a$  is congruent to  $b$  modulo  $r$ ,  
if  $a - b$  is divisible by  $r$ . We write

$$\boxed{a \equiv b \pmod{r}} . ”$$

**Exercise 1.** True or false:

- (1)  $22 \equiv 0 \pmod{5}$ .      (2)  $13 \equiv 3 \pmod{4}$ .      (3)  $100 \equiv 0 \pmod{25}$ .  
(4)  $17 \equiv 2 \pmod{5}$ .      (5)  $64 \equiv 4 \pmod{20}$ .      (6)  $121 \equiv 10 \pmod{37}$ .

[**Answers**]:      (1) False.      (2) False.      (3) True.  
(4) True.      (5) True.      (6) True.

Today I am going to rely on the notion of ‘congruence modulo 4’. Agree that

- 0, 4, 8, 12, 16, 20, 24, 28, ... are all congruent to 0 modulo 4,
- 1, 5, 9, 13, 17, 21, 25, 29, ... are all congruent to 1 modulo 4,
- 2, 6, 10, 14, 18, 22, 26, 30, ... are all congruent to 2 modulo 4,
- 3, 7, 11, 15, 19, 23, 27, 31, ... are all congruent to 3 modulo 4.

Now, with all that in mind, let’s recall what we saw last time (“Review of Lectures – XXXVI”):

**Summary.** Let  $x$  be an arbitrary real number. Then

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots,$$

and

$$\sin x = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots.$$

I said these can be the definition of  $\sin x$  and  $\cos x$ . Meanwhile, let's recall (from "Review of Lectures – XVIII"):

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots$$

The patterns in those three look alike. Is there a relation between

$$e^x, \quad \cos x \quad \text{and} \quad \sin x ?$$

If you just stay within the real numbers, you cannot find any relationship. As for this, remember that I briefly touched the subject of complex numbers, and more specifically the role of the unit imaginary number  $\sqrt{-1}$ , in "Review of Lectures – XV". An important role is played by not just  $\sqrt{-1}$  alone, but its powers. When it comes to the powers,  $\sqrt{-1}$  and  $-1$  are in sync with each other. Let's recall:

- **$(-1)$ -to-the-powers.** We have

$$\begin{aligned} (-1)^1 &= -1, \\ (-1)^2 &= 1, \\ (-1)^3 &= -1, \\ (-1)^4 &= 1, \\ (-1)^5 &= -1, \\ (-1)^6 &= 1, \\ (-1)^7 &= -1, \\ (-1)^8 &= 1, \\ \vdots & \quad \quad \quad \vdots \end{aligned}$$

In short,

$$\boxed{(-1)^n = \begin{cases} 1 & (\text{if } n \text{ is } \underline{\underline{\text{even}}}), \\ -1 & (\text{if } n \text{ is } \underline{\underline{\text{odd}}}). \end{cases}}$$

- **$(\sqrt{-1})$ -to-the-powers.** Here, we adopt the unit imaginary number

$$i = \sqrt{-1}.$$

This number is 'defined' as

$$i^2 = -1.$$

So,

$$i^1 = i,$$

$$i^2 = -1,$$

$$i^3 = -i,$$

$$i^4 = 1,$$

$$i^5 = i,$$

$$i^6 = -1,$$

$$i^7 = -i,$$

$$i^8 = 1,$$

$$i^9 = i,$$

$$i^{10} = -1,$$

$$i^{11} = -i,$$

$$i^{12} = 1,$$

⋮

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In short,

$$i^n = \begin{cases} 1 & (\text{if } n \equiv 0), \\ i & (\text{if } n \equiv 1), \\ -1 & (\text{if } n \equiv 2), \\ -i & (\text{if } n \equiv 3). \end{cases}$$

With that in mind, why don't we substitute  $x$  with  $ix$  in

$$\begin{aligned} e^x &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 \\ &+ \frac{1}{8!}x^8 + \frac{1}{9!}x^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}x^{11} + \frac{1}{12!}x^{12} + \frac{1}{13!}x^{13} + \frac{1}{14!}x^{14} + \frac{1}{15!}x^{15} \\ &+ \frac{1}{16!}x^{16} + \frac{1}{17!}x^{17} + \frac{1}{18!}x^{18} + \frac{1}{19!}x^{19} + \frac{1}{20!}x^{20} + \frac{1}{21!}x^{21} + \frac{1}{22!}x^{22} + \frac{1}{23!}x^{23} \\ &+ \frac{1}{24!}x^{24} + \frac{1}{25!}x^{25} + \frac{1}{26!}x^{26} + \frac{1}{27!}x^{27} + \frac{1}{28!}x^{28} + \frac{1}{29!}x^{29} + \frac{1}{30!}x^{30} + \frac{1}{31!}x^{31} \\ &+ \frac{1}{32!}x^{32} + \frac{1}{33!}x^{33} + \frac{1}{34!}x^{34} + \frac{1}{35!}x^{35} + \frac{1}{36!}x^{36} + \frac{1}{37!}x^{37} + \frac{1}{38!}x^{38} + \frac{1}{39!}x^{39} \\ &+ \dots \end{aligned}$$

The outcome is

$e^{ix}$

$$\begin{aligned} &= 1 + \frac{1}{1!}(ix) + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \frac{1}{5!}(ix)^5 + \frac{1}{6!}(ix)^6 + \frac{1}{7!}(ix)^7 \\ &+ \frac{1}{8!}(ix)^8 + \frac{1}{9!}(ix)^9 + \frac{1}{10!}(ix)^{10} + \frac{1}{11!}(ix)^{11} + \frac{1}{12!}(ix)^{12} + \frac{1}{13!}(ix)^{13} + \frac{1}{14!}(ix)^{14} + \frac{1}{15!}(ix)^{15} \\ &+ \frac{1}{16!}(ix)^{16} + \frac{1}{17!}(ix)^{17} + \frac{1}{18!}(ix)^{18} + \frac{1}{19!}(ix)^{19} + \frac{1}{20!}(ix)^{20} + \frac{1}{21!}(ix)^{21} + \frac{1}{22!}(ix)^{22} + \frac{1}{23!}(ix)^{23} \\ &+ \frac{1}{24!}(ix)^{24} + \frac{1}{25!}(ix)^{25} + \frac{1}{26!}(ix)^{26} + \frac{1}{27!}(ix)^{27} + \frac{1}{28!}(ix)^{28} + \frac{1}{29!}(ix)^{29} + \frac{1}{30!}(ix)^{30} + \frac{1}{31!}(ix)^{31} \\ &+ \frac{1}{32!}(ix)^{32} + \frac{1}{33!}(ix)^{33} + \frac{1}{34!}(ix)^{34} + \frac{1}{35!}(ix)^{35} + \frac{1}{36!}(ix)^{36} + \frac{1}{37!}(ix)^{37} + \frac{1}{38!}(ix)^{38} + \frac{1}{39!}(ix)^{39} \\ &+ \dots \\ &= 1 + \frac{1}{1!}ix + \frac{1}{2!}i^2x^2 + \frac{1}{3!}i^3x^3 + \frac{1}{4!}i^4x^4 + \frac{1}{5!}i^5x^5 + \frac{1}{6!}i^6x^6 + \frac{1}{7!}i^7x^7 \\ &+ \frac{1}{8!}i^8x^8 + \frac{1}{9!}i^9x^9 + \frac{1}{10!}i^{10}x^{10} + \frac{1}{11!}i^{11}x^{11} + \frac{1}{12!}i^{12}x^{12} + \frac{1}{13!}i^{13}x^{13} + \frac{1}{14!}i^{14}x^{14} + \frac{1}{15!}i^{15}x^{15} \\ &+ \frac{1}{16!}i^{16}x^{16} + \frac{1}{17!}i^{17}x^{17} + \frac{1}{18!}i^{18}x^{18} + \frac{1}{19!}i^{19}x^{19} + \frac{1}{20!}i^{20}x^{20} + \frac{1}{21!}i^{21}x^{21} + \frac{1}{22!}i^{22}x^{22} + \frac{1}{23!}i^{23}x^{23} \\ &+ \frac{1}{24!}i^{24}x^{24} + \frac{1}{25!}i^{25}x^{25} + \frac{1}{26!}i^{26}x^{26} + \frac{1}{27!}i^{27}x^{27} + \frac{1}{28!}i^{28}x^{28} + \frac{1}{29!}i^{29}x^{29} + \frac{1}{30!}i^{30}x^{30} + \frac{1}{31!}i^{31}x^{31} \\ &+ \frac{1}{32!}i^{32}x^{32} + \frac{1}{33!}i^{33}x^{33} + \frac{1}{34!}i^{34}x^{34} + \frac{1}{35!}i^{35}x^{35} + \frac{1}{36!}i^{36}x^{36} + \frac{1}{37!}i^{37}x^{37} + \frac{1}{38!}i^{38}x^{38} + \frac{1}{39!}i^{39}x^{39} \\ &+ \dots \end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{1}{1!}ix - \frac{1}{2!}x^2 - \frac{1}{3!}ix^3 + \frac{1}{4!}x^4 + \frac{1}{5!}ix^5 - \frac{1}{6!}x^6 - \frac{1}{7!}ix^7 \\
&+ \frac{1}{8!}x^8 + \frac{1}{9!}ix^9 - \frac{1}{10!}x^{10} - \frac{1}{11!}ix^{11} + \frac{1}{12!}x^{12} + \frac{1}{13!}ix^{13} - \frac{1}{14!}x^{14} - \frac{1}{15!}ix^{15} \\
&+ \frac{1}{16!}x^{16} + \frac{1}{17!}ix^{17} - \frac{1}{18!}x^{18} - \frac{1}{19!}ix^{19} + \frac{1}{20!}x^{20} + \frac{1}{21!}ix^{21} - \frac{1}{22!}x^{22} - \frac{1}{23!}ix^{23} \\
&+ \frac{1}{24!}x^{24} + \frac{1}{25!}ix^{25} - \frac{1}{26!}x^{26} - \frac{1}{27!}ix^{27} + \frac{1}{28!}x^{28} + \frac{1}{29!}ix^{29} - \frac{1}{30!}x^{30} - \frac{1}{31!}ix^{31} \\
&+ \frac{1}{32!}x^{32} + \frac{1}{33!}ix^{33} - \frac{1}{34!}x^{34} - \frac{1}{35!}ix^{35} + \frac{1}{36!}x^{36} + \frac{1}{37!}ix^{37} - \frac{1}{38!}x^{38} - \frac{1}{39!}ix^{39} \\
&+ \dots
\end{aligned}$$

Notice that at every other terms ' $i$ ' shows up. So, let's separate those terms with  $i$  and those without  $i$  from each other:

$$\begin{aligned}
e^{ix} &= \left( 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12} - \frac{1}{14!}x^{14} \right. \\
&+ \frac{1}{16!}x^{16} - \frac{1}{18!}x^{18} + \frac{1}{20!}x^{20} - \frac{1}{22!}x^{22} + \frac{1}{24!}x^{24} - \frac{1}{26!}x^{26} + \frac{1}{28!}x^{28} - \frac{1}{30!}x^{30} \\
&+ \left. \frac{1}{32!}x^{32} - \frac{1}{34!}x^{34} + \frac{1}{36!}x^{36} - \frac{1}{38!}x^{38} + \dots \right) \\
&+ i \left( \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \frac{1}{13!}x^{13} - \frac{1}{15!}x^{15} \right. \\
&+ \frac{1}{17!}x^{17} - \frac{1}{19!}x^{19} + \frac{1}{21!}x^{21} - \frac{1}{23!}x^{23} + \frac{1}{25!}x^{25} - \frac{1}{27!}x^{27} + \frac{1}{29!}x^{29} - \frac{1}{31!}x^{31} \\
&+ \left. \frac{1}{33!}x^{33} - \frac{1}{35!}x^{35} + \frac{1}{37!}x^{37} - \frac{1}{39!}x^{39} + \dots \right).
\end{aligned}$$

Can you identify the contents of the two big parentheses? Yes, they are exactly  $\cos x$  and  $\sin x$ . So, in short:

**Euler's formula.**

$$e^{ix} = (\cos x) + i(\sin x).$$

Now, this is not entirely absurd. On the contrary, this is actually very illuminating. Indeed, let

$$\alpha = e^{ix}, \quad \beta = e^{iy}.$$

Then

$$\begin{aligned} \alpha\beta &= e^{ix} e^{iy} \\ &= \left( (\cos x) + i(\sin x) \right) \left( (\cos y) + i(\sin y) \right). \end{aligned}$$

Let's expand this, taking into account  $i^2 = -1$ :

$$\begin{aligned} (*) \quad \alpha\beta &= \left[ (\cos x)(\cos y) - (\sin x)(\sin y) \right] \\ &\quad + i \left[ (\cos x)(\sin y) + (\sin x)(\cos y) \right]. \end{aligned}$$

And that was just simply the multiplication. But then suddenly let's compare this with the pair of formulas highlighted as 'Axiom 3' and 'Axiom 4' in "Review of Lectures - XXXIII" (I take the liberty to change the variables from  $\theta$  and  $\phi$  to  $x$  and  $y$ ):

**Axiom 3.**

$$\cos(x+y) = (\cos x)(\cos y) - (\sin x)(\sin y).$$

**Axiom 4.**

$$\sin(x+y) = (\sin x)(\cos y) + (\cos x)(\sin y).$$



You realize that the right-hand side of the identity (\*) on the past page is

$$\left( \cos (x+y) \right) + i \left( \sin (x+y) \right).$$

But wouldn't this exactly be  $e^{i(x+y)}$ ? Yes indeed. So, we have actually managed to prove

**Exponential Law – II.** Let  $x$  and  $y$  be real numbers. Then

$$e^{ix} e^{iy} = e^{i(x+y)}.$$

In case you are skeptical, this is indeed eligible to be called 'exponential law'. The original exponential law ('Rule II' in "Review of Lectures – XVIII") is

**Exponential Law (Original).** Let  $x$  and  $y$  be real numbers. Then

$$e^x e^y = e^{x+y}.$$

Nothing hinders us from viewing the above two as two offsprings of the same principle, namely, the exponential law. In other words, in retrospect, 'Axiom 3' and 'Axiom 4' combined was a disguised form of the exponential law. Or, at least that's one way to look at it. Actually, this is pretty standard, as in this is exactly how modern mathematicians look at 'sin' and 'cos'.