Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – XXXVI

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§36. Trigonometry – V.

We have introduced definite integrals. Today we start with a pair of basic rules, which are summarized in one term:

" linearity."

We use this term as in either one of the following ways (which are synonymous):

$$\text{“}\underline{\underline{\textit{the operator}}} \quad \int_{t=a}^{b} \Big(\ \bullet \ \Big) \ dt \quad \underline{\underline{\textit{is linear,}}}\text{”}$$

or,

$$\text{“} \int_{t=a}^{b} \Big(\ \bullet \ \Big) \ dt \quad \underline{\underline{\textit{is a linear operator.}}} \text{”}$$

- Below is what these sentences mean:

Rule 1.

$$\int_{t=a}^{b} \Big(f(t) + g(t) \Big) \ dt = \left(\int_{t=a}^{b} f(t) \, dt \right) + \left(\int_{t=a}^{b} g(t) \, dt \right).$$

Rule 2. If c is a constant (real number), then

$$\int_{t=a}^{b} c \cdot f(t) \, dt = c \left(\int_{t=a}^{b} f(t) \ dt \right).$$

The validity of these should be self-evident from 'Fundamental Theorem' (from the last lecture, "Review of Lectures – XXXV").

Example 1. $$\int_{t=0}^{1} \left(t^3 + t^2\right) dt = \left(\int_{t=0}^{1} t^3 \, dt \right) + \left(\int_{t=0}^{1} t^2 \, dt \right)$$

$$= \left[\frac{1}{4} t^4 \right]_{t=0}^{1} + \left[\frac{1}{3} t^3 \right]_{t=0}^{1}$$

$$= \frac{1}{4} \cdot 1^4 + \frac{1}{3} \cdot 1^3 = \frac{7}{12}.$$

⋆ You can do it the following way:

$$\int_{t=0}^{1} \left(t^3 + t^2\right) dt = \left[\frac{1}{4} t^4 + \frac{1}{3} t^3 \right]_{t=0}^{1}$$

$$= \frac{1}{4} \cdot 1^4 + \frac{1}{3} \cdot 1^3 = \frac{7}{12}.$$

⋆ Two methods have produced the same answer. That was pre-assured by Rule 1.

Example 2. $$\int_{t=-1}^{4} \left(t^4 + 1\right) dt = \left(\int_{t=-1}^{4} t^4 \, dt \right) + \left(\int_{t=-1}^{4} 1 \, dt \right)$$

$$= \left[\frac{1}{5} t^5 \right]_{t=-1}^{4} + \left[t \right]_{t=-1}^{4}$$

$$= \left(\frac{1}{5} \cdot 4^5 - \frac{1}{5} \cdot \left(-1\right)^5 \right) + \left(4 - \left(-1\right)\right)$$

$$= 210.$$

⋆ You can do it the following way:

$$\int_{t=-1}^{4} \left(t^4 + 1\right)\, dt = \left[\frac{1}{5}t^5 + t\right]_{t=-1}^{4}$$

$$= \left(\frac{1}{5}\cdot 4^5 + 4\right) - \left(\frac{1}{5}\cdot\left(-1\right)^5 + \left(-1\right)\right)$$

$$= 210.$$

⋆ Two methods have produced the same answer. That was pre-assured by Rule 1.

Example 3.

$$\int_{t=0}^{2} 2t^3\, dt = 2\left(\int_{t=0}^{2} t^3\, dt\right)$$

$$= 2 \cdot \left[\frac{1}{4}t^4\right]_{t=0}^{2}$$

$$= 2 \cdot \left(\frac{1}{4}\cdot 2^4\right)$$

$$= 8.$$

⋆ You can do it the following way:

$$\int_{t=0}^{2} 2t^3\, dt = \left[\frac{1}{2}t^4\right]_{t=0}^{2}$$

$$= \frac{1}{2}\cdot 2^4$$

$$= 8.$$

⋆ Two methods have produced the same answer. That was pre-assured by Rule 2.

Example 4. $$\int_{t=0}^{\frac{\pi}{3}} \Big(2\cos t \;+\; 3\sin t\Big)\; dt$$

$$= \; 2\left(\int_{t=0}^{\frac{\pi}{3}} \cos t\; dt\right) \;+\; 3\left(\int_{t=0}^{\frac{\pi}{3}} \sin t\; dt\right)$$

$$= \; 2 \cdot \Big[\; \sin t \;\Big]_{t=0}^{\frac{\pi}{3}} \;+\; 3 \cdot \Big[\; -\cos t \;\Big]_{t=0}^{\frac{\pi}{3}}$$

$$= \; 2 \cdot \left(\left(\sin \frac{\pi}{3}\right) - \Big(\sin 0\Big)\right) + 3 \cdot \left(\left(-\cos\frac{\pi}{3}\right) - \Big(-\cos 0\Big)\right)$$

$$= \; 2 \cdot \left(\frac{\sqrt{3}}{2} - 0\right) + 3 \cdot \left(\left(-\frac{1}{2}\right) - \Big(-1\Big)\right)$$

$$= \; \sqrt{3} + \frac{3}{2}.$$

⋆ You can do it the following way:

$$\int_{t=0}^{\frac{\pi}{3}} \Big(2\cos t \;+\; 3\sin t\Big)\; dt$$

$$= \; \Big[\; 2\sin t \;-\; 3\cos t \;\Big]_{t=0}^{\frac{\pi}{3}}$$

$$= \; \left(2\left(\sin\frac{\pi}{3}\right) - 3\left(\cos\frac{\pi}{3}\right)\right) - \left(2\Big(\sin 0\Big) - 3\Big(\cos 0\Big)\right)$$

$$= \; \left(2 \cdot \frac{\sqrt{3}}{2} - 3 \cdot \frac{1}{2}\right) - \Big(2 \cdot 0 - 3 \cdot 1\Big) \;=\; \sqrt{3} + \frac{3}{2}.$$

⋆ Two methods have produced the same answer. That was pre-assured by Rule 1 and Rule 2.

Exercise 1. Evaluate

(1) $$\int_{t=3}^{5} \left(t^2 + 2t\right)\, dt.$$ (2) $$\int_{t=0}^{\frac{3}{2}} \left(4t^3 - 3t^2\right)\, dt.$$

(3) $$\int_{t=0}^{1} \left(t^3 - \frac{3}{2}t^2 + \frac{1}{2}t\right)\, dt.$$

(4) $$\int_{t=-1}^{0} \frac{\left(t+1\right)\left(t+2\right)\left(t+3\right)}{3!}\, dt.$$

(5) $$\int_{t=\frac{\pi}{6}}^{\frac{\pi}{3}} \left(3\cos t - 4\sin t\right)\, dt.$$

(6) $$\int_{t=0}^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right)\, dt.$$

[**Answers**]: (1) $\frac{146}{3}$. (2) $\frac{27}{16}$. (3) $\frac{9}{64}$.

(4) $\frac{3}{8}$. (5) $\frac{1-\sqrt{3}}{2}$. (6) $\sqrt{2}$.

- Next, I show you something which looks innocuous but is extremely important:

Formula 1. Suppose

$$f(t) \le g(t)$$

holds whenever $a \le t \le b$. Then

$$\int_{t=a}^{b} f(t)\,dt \le \int_{t=a}^{b} g(t)\,dt.$$

⋆ Let's apply this formula to the following situation: We have agreed that

$$\sin t \le t$$

holds whenever $0 \le t$. So, choose an arbitrary positive real number x, and we have

$$\int_{t=0}^{x} \sin t\,dt \le \int_{t=0}^{x} t\,dt,$$

that is,

$$1 - \cos x \le \frac{1}{2}x^2.$$

Shift appropriate terms and obtain

$$1 - \frac{1}{2}x^2 \le \cos x.$$

Apply Formula 1 again:

$$\int_{t=0}^{x} \left(1 - \frac{1}{2}t^2\right) dt \le \int_{t=0}^{x} \cos t\,dt,$$

that is,

$$x - \frac{1}{2 \cdot 3}x^3 \le \sin x.$$

Apply Formula 1 again:

$$\int_{t=0}^{x} \left(t - \frac{1}{2 \cdot 3} t^3 \right) dt \leq \int_{t=0}^{x} \sin t \, dt,$$

that is,

$$\frac{1}{2} x^2 - \frac{1}{2 \cdot 3 \cdot 4} x^4 \leq 1 - \cos x.$$

Shift appropriate terms and obtain

$$\cos x \leq 1 - \frac{1}{2} x^2 + \frac{1}{2 \cdot 3 \cdot 4} x^4.$$

Apply Formula 1 again:

$$\int_{t=0}^{x} \cos t \, dt \leq \int_{t=0}^{x} \left(1 - \frac{1}{2} t^2 + \frac{1}{2 \cdot 3 \cdot 4} t^4 \right) dt,$$

that is,

$$\sin x \leq x - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} x^5.$$

Apply Formula 1 again:

$$\int_{t=0}^{x} \sin t \, dt \leq \int_{t=0}^{x} \left(t - \frac{1}{2 \cdot 3} t^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} t^5 \right) dt,$$

that is,

$$1 - \cos x \leq \frac{1}{2} x^2 - \frac{1}{2 \cdot 3 \cdot 4} x^4 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} x^6.$$

Shift appropriate terms and obtain

$$1 - \frac{1}{2} x^2 + \frac{1}{2 \cdot 3 \cdot 4} x^4 - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} x^6 \leq \cos x.$$

Now, let's summarize what we have got so far:

$$
\begin{aligned}
\sin x &\le x, \\
1 - \frac{1}{2}x^2 &\le \cos x, \\
x - \frac{1}{2 \cdot 3}x^3 &\le \sin x, \\
\cos x &\le 1 - \frac{1}{2}x^2 + \frac{1}{2 \cdot 3 \cdot 4}x^4, \\
\sin x &\le x - \frac{1}{2 \cdot 3}x^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}x^5, \\
1 - \frac{1}{2}x^2 + \frac{1}{2 \cdot 3 \cdot 4}x^4 - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}x^6 &\le \cos x, \\
&\vdots
\end{aligned}
$$

There is a pattern here. First of all, notice that the denominators are all factorial numbers. So why don't we rewrite these as

$$
\begin{aligned}
\sin x &\le \frac{1}{1!}x, \\
1 - \frac{1}{2!}x^2 &\le \cos x, \\
\frac{1}{1!}x - \frac{1}{3!}x^3 &\le \sin x, \\
\cos x &\le 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4, \\
\sin x &\le \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5, \\
1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 &\le \cos x, \\
&\vdots
\end{aligned}
$$

Let's separate those lines involving $\cos x$ from those involving $\sin x$:

$$1 - \frac{1}{2!}x^2 \leq \cos x,$$

$$\cos x \leq 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4,$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \leq \cos x,$$

$$\vdots$$

and also

$$\sin x \leq \frac{1}{1!}x,$$

$$\frac{1}{1!}x - \frac{1}{3!}x^3 \leq \sin x,$$

$$\sin x \leq \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5,$$

$$\vdots$$

⋆ You agree that, if we continue the procedure, then we will get the following:

○ (C-1)

$$1 - \frac{1}{2!}x^2 \leq \cos x,$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \leq \cos x,$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} \leq \cos x,$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12} - \frac{1}{14!}x^{14} \leq \cos x,$$

$$\vdots$$

- (C-2)

$$\cos x \;\le\; 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4,$$

$$\cos x \;\le\; 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8,$$

$$\cos x \;\le\; 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12},$$

$$\cos x \;\le\; 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12} - \frac{1}{14!}x^{14} + \frac{1}{16!}x^{16},$$

$$\vdots$$

- (S-1)

$$\frac{1}{1!}x - \frac{1}{3!}x^3 \;\le\; \sin x,$$

$$\frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \;\le\; \sin x,$$

$$\frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} \;\le\; \sin x,$$

$$\frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \frac{1}{13!}x^{13} - \frac{1}{15!}x^{15} \;\le\; \sin x,$$

$$\vdots$$

- (S-2)

$$\sin x \;\le\; 1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5,$$

$$\sin x \;\le\; 1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9,$$

$$\sin x \;\le\; 1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \frac{1}{13!}x^{13},$$

$$\vdots$$

$\star$ More generally, for an arbitrary positive integer n, and for an arbitrary real number x with $x > 0$:

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots - \frac{1}{\big(4n-2\big)!}x^{4n-2} \le \cos x.$$

$$\cos x \le 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots - \frac{1}{\big(4n-2\big)!}x^{4n-2} + \frac{1}{\big(4n\big)!}x^{4n}.$$

$$\frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots - \frac{1}{\big(4n-1\big)!}x^{4n-1} \le \sin x.$$

$$\sin x \le \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots - \frac{1}{\big(4n-1\big)!}x^{4n-1} + \frac{1}{\big(4n+1\big)!}x^{4n+1}.$$

What do these entail? Yes. Let's consider a 'virtual' infinite sum

$$C = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots,$$

$$S = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots,$$

Truncate these:

$$\begin{aligned}
C_0 &= 1, \\
C_2 &= 1 - \frac{1}{2!}x^2, \\
C_4 &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4, \\
C_6 &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6, \\
C_8 &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8, \\
&\vdots
\end{aligned}$$

Also

$$
\begin{aligned}
S_1 &= \frac{1}{1!}x, \\
S_3 &= \frac{1}{1!}x - \frac{1}{3!}x^3, \\
S_5 &= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5, \\
S_7 &= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7, \\
S_9 &= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9,
\end{aligned}
$$

Then

- ○ the distance between $\cos x$ and C_0 is at most $\frac{1}{2!}x^2$,
- ○ the distance between $\cos x$ and C_2 is at most $\frac{1}{4!}x^4$,
- ○ the distance between $\cos x$ and C_4 is at most $\frac{1}{6!}x^6$,
- ○ the distance between $\cos x$ and C_8 is at most $\frac{1}{8!}x^8$,

⋮

Also

- ○ the distance between $\sin x$ and S_1 is at most $\frac{1}{3!}x^3$,
- ○ the distance between $\sin x$ and S_3 is at most $\frac{1}{5!}x^5$,
- ○ the distance between $\sin x$ and S_5 is at most $\frac{1}{7!}x^7$,
- ○ the distance between $\sin x$ and S_7 is at most $\frac{1}{9!}x^9$,

⋮

Now, here is the last clue:

Formula 2. Let x be an arbitrary constant real number $\big($where 'constant' meanas it does not depend on $n\big)$. Then

$$\boxed{\lim_{n\to\infty} \frac{x^{2n}}{\big(2n\big)!} = 0}\,, \qquad \boxed{\lim_{n\to\infty} \frac{x^{2n+1}}{\big(2n+1\big)!} = 0}\,.$$

Hence

- ∘ the distance between $\cos x$ and C_{2n} approaches to 0 as $n \longrightarrow \infty$,
- ∘ the distance between $\sin x$ and S_{2n+1} approaches to 0 as $n \longrightarrow \infty$.

Hence the values of the virtual infinite sums

$$C = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots,$$

and

$$S = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots$$

both indeed exist, for an arbitrary positive real number x, and they equal $\cos x$, and $\sin x$, respectively. To repeat:

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots ,$$

and

$$\sin x = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots .$$

Now, in the above, throughout we have assumed $x > 0$. The fact of the matter is, the same are true for all real numbers x. To summarize:

Summary. Let x be an arbitrary real number. Then

$$\boxed{\cos x \;=\; 1 \;-\; \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots} \;,$$

and

$$\boxed{\sin x \;=\; \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots} \;.$$

⋆ You can regard these as the definition of $\sin x$ and $\cos x$. For example,

$$\cos 0 \;=\; 1 \qquad \text{and} \qquad \sin 0 \;=\; 0$$

are immediately retrieved from these. Also, by doing a term-wise differentiation, you rather instantaneously see the following:

Formula 3.

$$\boxed{\frac{d}{dx}\cos x \;=\; -\sin x} \;, \qquad \text{and} \qquad \boxed{\frac{d}{dx}\sin x \;=\; \cos x} \;.$$

Agree that Formula 3 is indeed a paraphrase of the following 'Quick Facts' from the last lecture ("Review of Lectures – XXXV"):

Quick Facts.

(1) An antiderivative of $\boxed{\cos x}$ is $\boxed{\sin x}$.

(2) An antiderivative of $\boxed{\sin x}$ is $\boxed{-\cos x}$.