Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XXXV

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Instructor: Yasuyuki Kachi

Line #: 52920.

§35. Trigonometry – IV. Definite integrals.

A quick review of what we have worked out two lectures ago in "Review of Lectures – XXXIII"': :

Axiom 1.	$\cos\Big(\theta-\phi\Big) = \Big(\cos\theta\Big)\Big(\cos\phi\Big) + \Big(\sin\theta\Big)\Big(\sin\phi\Big)$.
Axiom 2.	$\sin\Big(\theta-\phi\Big) = \Big(\sin\theta\Big)\Big(\cos\phi\Big) - \Big(\cos\theta\Big)\Big(\sin\phi\Big)$.
Axiom 3.	$\cos\Big(\theta+\phi\Big) = \Big(\cos\theta\Big)\Big(\cos\phi\Big) - \Big(\sin\theta\Big)\Big(\sin\phi\Big)$.
Axiom 4.	$\sin\Big(\theta+\phi\Big) = \Big(\sin\theta\Big)\Big(\cos\phi\Big) + \Big(\cos\theta\Big)\Big(\sin\phi\Big)$.

Double angle formula for cos.

$$\cos\Big(2\theta\Big) = \Big(\cos\theta\Big)^2 - \Big(\sin\theta\Big)^2 .$$

Double angle formula for cos – version 2.

$$\cos\Big(2\theta\Big) = 2\Big(\cos\theta\Big)^2 - 1 .$$

Double angle formula for cos – version 3.

$$\cos\Big(2\theta\Big) = 1 - 2\Big(\sin\theta\Big)^2 .$$

Double angle formula for sin.

$$\boxed{\sin\Big(2\theta\Big) \;=\; 2\Big(\cos\theta\Big)\Big(\sin\theta\Big)}\,.$$

Formula A. $$\boxed{2\Big(\cos\theta\Big)\Big(\cos\phi\Big) \;=\; \cos\Big(\theta-\phi\Big) + \cos\Big(\theta+\phi\Big)}\,.$$

Formula B. $$\boxed{2\Big(\sin\theta\Big)\Big(\sin\phi\Big) \;=\; \cos\Big(\theta-\phi\Big) - \cos\Big(\theta+\phi\Big)}\,.$$

Formula C. $$\boxed{2\Big(\sin\theta\Big)\Big(\cos\phi\Big) \;=\; \sin\Big(\theta-\phi\Big) + \sin\Big(\theta+\phi\Big)}\,.$$

Formula D. $$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \;=\; 1.}\,.$$

- Next, let's recall from the last lecture ("Review of Lectures – XXXIV")

Notation. For a given $f(x)$, the notation $M_n(f)(x)$ stands for the mean of

$$f\Big(\frac{x}{n}\Big),\ f\Big(\frac{2x}{n}\Big),\ f\Big(\frac{3x}{n}\Big),\ \cdots,\ f\Big(\frac{nx}{n}\Big).$$

Also, the notation $M(f)(x)$ stands for the limit

$$\lim_{n \to \infty} M_n(f)(x).$$

Formula. Let n be a positive integer; $n = 1, 2, 3, 4, \cdots$. Let

$$f(x) \;=\; x^n.$$

Then we have

$$\boxed{M(f)(x) = \frac{1}{n+1}x^n}\,.$$

- Today's first goal is to find $M(f)(x)$ and $M(g)(x)$ where

$$f(x) = \cos x, \qquad \text{and} \qquad g(x) = \sin x.$$

- As a first step, we consider

(1) $\cos \frac{x}{1}$,

(2) $\cos \frac{x}{2} + \cos \frac{2x}{2}$,

(3) $\cos \frac{x}{3} + \cos \frac{2x}{3} + \cos \frac{3x}{3}$,

(4) $\cos \frac{x}{4} + \cos \frac{2x}{4} + \cos \frac{3x}{4} + \cos \frac{4x}{4}$,

(5) $\cos \frac{x}{5} + \cos \frac{2x}{5} + \cos \frac{3x}{5} + \cos \frac{4x}{5} + \cos \frac{5x}{5}$,

(6) $\cos \frac{x}{6} + \cos \frac{2x}{6} + \cos \frac{3x}{6} + \cos \frac{4x}{6} + \cos \frac{5x}{6} + \cos \frac{6x}{6}$,

$\vdots$

Can we simplify these? Let's try (5). Once we have the answer for (5), we will know how to do others, as they all hold the same patterns.

- **Trick.** By Formula C,

$$\left(\sin \frac{x}{10}\right)\left(\cos \frac{x}{5}\right) = \sin\left(\frac{x}{10} - \frac{x}{5}\right) + \sin\left(\frac{x}{10} + \frac{x}{5}\right),$$

$$\left(\sin \frac{x}{10}\right)\left(\cos \frac{2x}{5}\right) = \sin\left(\frac{x}{10} - \frac{2x}{5}\right) + \sin\left(\frac{x}{10} + \frac{2x}{5}\right),$$

$$\left(\sin \frac{x}{10}\right)\left(\cos \frac{3x}{5}\right) = \sin\left(\frac{x}{10} - \frac{3x}{5}\right) + \sin\left(\frac{x}{10} + \frac{3x}{5}\right),$$

$$\left(\sin \frac{x}{10}\right)\left(\cos \frac{4x}{5}\right) = \sin\left(\frac{x}{10} - \frac{4x}{5}\right) + \sin\left(\frac{x}{10} + \frac{4x}{5}\right),$$

$$\left(\sin \frac{x}{10}\right)\left(\cos \frac{5x}{5}\right) = \sin\left(\frac{x}{10} - \frac{5x}{5}\right) + \sin\left(\frac{x}{10} + \frac{5x}{5}\right).$$

In each of the five equations, what's inside the parentheses on the right-hand side can be easily calculated, so

$$\begin{array}{rrcl}
& \left(\sin\dfrac{x}{10}\right)\left(\cos\dfrac{x}{5}\right) & = & -\left(\sin\dfrac{x}{10}\right)+\left(\sin\dfrac{3x}{10}\right), \\
& \left(\sin\dfrac{x}{10}\right)\left(\cos\dfrac{2x}{5}\right) & = & -\left(\sin\dfrac{3x}{10}\right)+\left(\sin\dfrac{5x}{10}\right), \\
& \left(\sin\dfrac{x}{10}\right)\left(\cos\dfrac{3x}{5}\right) & = & -\left(\sin\dfrac{5x}{10}\right)+\left(\sin\dfrac{7x}{10}\right), \\
& \left(\sin\dfrac{x}{10}\right)\left(\cos\dfrac{4x}{5}\right) & = & -\left(\sin\dfrac{7x}{10}\right)+\left(\sin\dfrac{9x}{10}\right), \\
& \left(\sin\dfrac{x}{10}\right)\left(\cos\dfrac{5x}{5}\right) & = & -\left(\sin\dfrac{9x}{10}\right)+\left(\sin\dfrac{11x}{10}\right). \\
+) & \hline
\end{array}$$

$$\left(\sin\frac{x}{10}\right)\Big(\text{the quantity (5)}\Big) = -\left(\sin\frac{x}{10}\right)+\left(\sin\frac{11x}{10}\right).$$

Here, the right-hand side

$$-\left(\sin\frac{x}{10}\right)+\left(\sin\frac{11x}{10}\right)$$

can be rewritten as

$$\left(\sin\frac{5x}{10}\right)\left(\cos\frac{6x}{10}\right)$$

$\big($reverse application of Formula C$\big)$, that is,

$$\left(\sin\frac{x}{2}\right)\left(\cos\frac{6x}{10}\right).$$

Thus

$$\left(\sin\frac{x}{10}\right)\Big(\text{the quantity (5)}\Big) \;=\; \left(\sin\frac{x}{2}\right)\left(\cos\frac{6x}{10}\right).$$

Divide the both sides by $\left(\sin\dfrac{x}{10}\right)$ and obtain

$$\Big(\text{the quantity (5)}\Big) \;=\; \frac{\left(\sin\dfrac{x}{2}\right)\left(\cos\dfrac{6x}{10}\right)}{\left(\sin\dfrac{x}{10}\right)}.$$

This way we have successfully simplified the quantity (5) in the list below $\big($duplicate of page 3$\big)$:

(1) $\cos\dfrac{x}{1}$,

(2) $\cos\dfrac{x}{2}+\cos\dfrac{2x}{2}$,

(3) $\cos\dfrac{x}{3}+\cos\dfrac{2x}{3}+\cos\dfrac{3x}{3}$,

(4) $\cos\dfrac{x}{4}+\cos\dfrac{2x}{4}+\cos\dfrac{3x}{4}+\cos\dfrac{4x}{4}$,

(5) $\cos\dfrac{x}{5}+\cos\dfrac{2x}{5}+\cos\dfrac{3x}{5}+\cos\dfrac{4x}{5}+\cos\dfrac{5x}{5}$,

(6) $\cos\dfrac{x}{6}+\cos\dfrac{2x}{6}+\cos\dfrac{3x}{6}+\cos\dfrac{4x}{6}+\cos\dfrac{5x}{6}+\cos\dfrac{6x}{6}$,

$\vdots$

The rest is similar:

$$\left(\text{the quantity (1)}\right) = \frac{\left(\sin\frac{x}{2}\right)\left(\cos\frac{2x}{2}\right)}{\left(\sin\frac{x}{2}\right)}.$$

$$\left(\text{the quantity (2)}\right) = \frac{\left(\sin\frac{x}{2}\right)\left(\cos\frac{3x}{4}\right)}{\left(\sin\frac{x}{4}\right)}.$$

$$\left(\text{the quantity (3)}\right) = \frac{\left(\sin\frac{x}{2}\right)\left(\cos\frac{4x}{6}\right)}{\left(\sin\frac{x}{6}\right)}.$$

$$\left(\text{the quantity (4)}\right) = \frac{\left(\sin\frac{x}{2}\right)\left(\cos\frac{5x}{8}\right)}{\left(\sin\frac{x}{8}\right)}.$$

$$\left(\text{the quantity (5)}\right) = \frac{\left(\sin\frac{x}{2}\right)\left(\cos\frac{6x}{10}\right)}{\left(\sin\frac{x}{10}\right)}.$$

$$\left(\text{the quantity (6)}\right) = \frac{\left(\sin\frac{x}{2}\right)\left(\cos\frac{7x}{12}\right)}{\left(\sin\frac{x}{12}\right)}.$$

Recognize the patterns, and we may concocut a formula for line (n):

$$\left(\text{the quantity } (n)\right) = \frac{\left(\sin\frac{x}{2}\right)\left(\cos\left(\frac{x}{2}+\frac{x}{2n}\right)\right)}{\left(\sin\frac{x}{2n}\right)}.$$

Summary.

$$\cos\,\frac{x}{n}\;+\;\cos\,\frac{2x}{n}\;+\;\cos\,\frac{3x}{n}\;+\;\cos\,\frac{4x}{n}\;+\;\cdots\;+\;\cos\,\frac{nx}{n}$$

$$=\;\frac{\left(\sin\,\frac{x}{2}\right)\left(\cos\,\left(\frac{x}{2}+\frac{x}{2n}\right)\right)}{\left(\sin\,\frac{x}{2n}\right)}.$$

- Hence, for

$$f\,(x)\;=\;\cos\,x,$$

the mean $M_n\,(f)\,(x)$ of

$$f\Big(\frac{x}{n}\Big),\; f\Big(\frac{2x}{n}\Big),\; f\Big(\frac{3x}{n}\Big),\;\cdots,\; f\left(\frac{nx}{n}\right)$$

is given by

$$M_n\,(f)\,(x)\;=\;\frac{\left(\sin\,\frac{x}{2}\right)\left(\cos\,\left(\frac{x}{2}+\frac{x}{2n}\right)\right)}{n\left(\sin\,\frac{x}{2n}\right)}.$$

We are interested in the limit of this as $n\longrightarrow\infty$. As for this, observe that, as $n\longrightarrow\infty$ the second factor in the numeraetor $\cos\left(\frac{x}{2}+\frac{x}{2n}\right)$ clearly approaches to $\cos\,\frac{x}{2}$. Since the first factor in the numerator $\sin\,\frac{x}{2}$ does not involve n, thus the limit

$$M\,(f)\,(x)\;=\;\lim_{n\,\longrightarrow\,\infty}\,M_n\,(f)\,(x)$$

simply equals

$$\left(\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}\right)\cdot\lim_{n\longrightarrow\infty}\frac{1}{n\left(\sin\frac{x}{2n}\right)}.$$

So,

$$(*)\qquad \boxed{M\left(f\right)\left(x\right) \;=\; \left(\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}\right)\cdot\lim_{n\longrightarrow\infty}\frac{1}{n\left(\sin\frac{x}{2n}\right)}}\;.$$

We need to compute this limit. For that matter, it will be beneficial to artificially rewrite what's inside the limit symbol

$$\frac{1}{n\left(\sin\frac{x}{2n}\right)}$$

as

$$\frac{2}{x}\cdot\frac{\frac{x}{2n}}{\left(\sin\frac{x}{2n}\right)}.$$

The benefit of that is clearly $\frac{x}{2n}\longrightarrow 0$ as $n\longrightarrow\infty$. So Formula D $\big($see page 2$\big)$ is applicable.

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\frac{1}{n\left(\sin\frac{x}{2n}\right)} &= \lim_{n\longrightarrow\infty}\frac{2}{x}\cdot\frac{\frac{x}{2n}}{\left(\sin\frac{x}{2n}\right)}\\
&= \frac{2}{x}\cdot\lim_{n\longrightarrow\infty}\frac{\frac{x}{2n}}{\left(\sin\frac{x}{2n}\right)}\\
&= \frac{2}{x}\cdot\lim_{\theta\longrightarrow 0}\frac{\theta}{\sin\,\theta}.
\end{aligned}$$

By Formula D $\big($see page 2$\big)$, this last limit is 1. So, in short

$$\lim_{n \to \infty} \frac{1}{n \left(\sin \dfrac{x}{2n} \right)} = \frac{2}{x}.$$

Incorporate this into $(*)$ in the previous page, and conclude

$$M\left(f\right)\left(x\right) = \frac{2}{x} \cdot \left(\sin \frac{x}{2} \right) \left(\cos \frac{x}{2} \right).$$

By 'Double Angle Formula' $\big($for 'sin'; in page 2$\big)$, we may simplify this as $\dfrac{1}{x} \sin x$, or the same $\dfrac{\sin x}{x}$. Thus we arrive at the conclusion:

Conclusion. For

$$f\left(x\right) = \cos x,$$

we have

$$\begin{cases} M_n\left(f\right)\left(x\right) = \dfrac{\left(\sin \dfrac{x}{2} \right) \left(\cos \left(\dfrac{x}{2} + \dfrac{x}{2n} \right) \right)}{n \left(\sin \dfrac{x}{2n} \right)}, \\ \\ M\left(f\right)\left(x\right) = \dfrac{\sin x}{x}. \end{cases}$$

- Out next job is to do the same for

$$g\left(x\right) = \sin x.$$

This is similar. Namely, first consider

$$(1) \qquad \sin\frac{x}{1},$$

$$(2) \qquad \sin\frac{x}{2} + \sin\frac{2x}{2},$$

$$(3) \qquad \sin\frac{x}{3} + \sin\frac{2x}{3} + \sin\frac{3x}{3},$$

$$(4) \qquad \sin\frac{x}{4} + \sin\frac{2x}{4} + \sin\frac{3x}{4} + \sin\frac{4x}{4},$$

$$(5) \qquad \sin\frac{x}{5} + \sin\frac{2x}{5} + \sin\frac{3x}{5} + \sin\frac{4x}{5} + \sin\frac{5x}{5},$$

$$(6) \qquad \sin\frac{x}{6} + \sin\frac{2x}{6} + \sin\frac{3x}{6} + \sin\frac{4x}{6} + \sin\frac{5x}{6} + \sin\frac{6x}{6},$$

$$\vdots$$

For example, (5) is taken care of by Formula B (in page 2), as follows:

$$\begin{aligned}
\left(\sin\frac{x}{10}\right)\left(\sin\frac{x}{5}\right) &= \left(\cos\frac{x}{10}\right) - \left(\cos\frac{3x}{10}\right),\\
\left(\sin\frac{x}{10}\right)\left(\sin\frac{2x}{5}\right) &= \left(\cos\frac{3x}{10}\right) - \left(\cos\frac{5x}{10}\right),\\
\left(\sin\frac{x}{10}\right)\left(\sin\frac{3x}{5}\right) &= \left(\cos\frac{5x}{10}\right) - \left(\cos\frac{7x}{10}\right),\\
\left(\sin\frac{x}{10}\right)\left(\sin\frac{4x}{5}\right) &= \left(\cos\frac{7x}{10}\right) - \left(\cos\frac{9x}{10}\right),\\
+)\quad \left(\sin\frac{x}{10}\right)\left(\sin\frac{5x}{5}\right) &= \left(\cos\frac{9x}{10}\right) - \left(\cos\frac{11x}{10}\right).\\
\hline
\left(\sin\frac{x}{10}\right)\left(\text{the quantity (5)}\right) &= \left(\cos\frac{x}{10}\right) - \left(\cos\frac{11x}{10}\right).
\end{aligned}$$

Here, the right-hand side

$$\left(\cos\frac{x}{10}\right) - \left(\cos\frac{11x}{10}\right)$$

can be rewritten as

$$\left(\sin\frac{5x}{10}\right)\left(\sin\frac{6x}{10}\right)$$

$\big($reverse application of Formula B$\big)$, that is,

$$\left(\sin\frac{x}{2}\right)\left(\sin\frac{6x}{10}\right).$$

Thus

$$\left(\sin\frac{x}{10}\right)\Big(\text{the quantity } (5)\Big) \;=\; \left(\sin\frac{x}{2}\right)\left(\sin\frac{6x}{10}\right).$$

Divide the both sides by $\left(\sin\dfrac{x}{10}\right)$ and obtain

$$\Big(\text{the quantity } (5)\Big) \;=\; \frac{\left(\sin\dfrac{x}{2}\right)\left(\sin\dfrac{6x}{10}\right)}{\left(\sin\dfrac{x}{10}\right)}.$$

More generally,

$$\Big(\text{the quantity } (n)\Big) \;=\; \frac{\left(\sin\dfrac{x}{2}\right)\left(\sin\left(\dfrac{x}{2}+\dfrac{x}{2n}\right)\right)}{\left(\sin\dfrac{x}{2n}\right)}.$$

Summary.

$$\sin\frac{x}{n} + \sin\frac{2x}{n} + \sin\frac{3x}{n} + \sin\frac{4x}{n} + \cdots + \sin\frac{nx}{n}$$

$$= \frac{\left(\sin\frac{x}{2}\right)\left(\sin\left(\frac{x}{2}+\frac{x}{2n}\right)\right)}{\left(\sin\frac{x}{2n}\right)}.$$

- Hence, for

$$g(x) = \sin x,$$

the mean $M_n(g)(x)$ of

$$g\left(\frac{x}{n}\right),\ g\left(\frac{2x}{n}\right),\ g\left(\frac{3x}{n}\right),\ \cdots,\ g\left(\frac{nx}{n}\right)$$

is given by

$$M_n(g)(x) = \frac{\left(\sin\frac{x}{2}\right)\left(\sin\left(\frac{x}{2}+\frac{x}{2n}\right)\right)}{n\left(\sin\frac{x}{2n}\right)}.$$

Finally, as for the limit

$$M(g)(x) = \lim_{n\longrightarrow\infty} M_n(g)(x),$$

observe that, as $n \longrightarrow \infty$ the second factor in the numeraetor $\sin\left(\frac{x}{2}+\frac{x}{2n}\right)$ clearly approaches to $\sin\frac{x}{2}$, whereas the first factor in the numerator $\sin\frac{x}{2}$ does not involve n. Thus

$$\lim_{n \to \infty} M_n(g)(x) = \left(\sin \frac{x}{2}\right)\left(\sin \frac{x}{2}\right) \cdot \lim_{n \to \infty} \frac{1}{n\left(\sin \frac{x}{2n}\right)}$$

$$= \left(\sin \frac{x}{2}\right)^2 \cdot \frac{2}{x}$$

$$\left(\text{since} \quad \lim_{n \to \infty} \frac{1}{n\left(\sin \frac{x}{2n}\right)} = \frac{2}{x}, \quad \text{as worked out in page 8}\right)$$

$$= \frac{2\left(\sin \frac{x}{2}\right)^2}{x}$$

$$= \frac{1 - \cos x}{x}.$$

Here, the last equality is thanks to 'Double Angle Formula for cos – version 3 (page 1).

Conclusion. For

$$g(x) = \sin x,$$

we have

$$\begin{cases} M_n(g)(x) = \dfrac{\left(\sin \frac{x}{2}\right)\left(\sin\left(\frac{x}{2} + \frac{x}{2n}\right)\right)}{n\left(\sin \frac{x}{2n}\right)}, \\ \\ M(g)(x) = \dfrac{1 - \cos x}{x}. \end{cases}$$

- **Definite integrals.**

Back in "Review of Lectures – XXVIII", we introduced "indefinite integrals". Today we talk about

" definite integral. "

The notation is

$$\int_{t=0}^{x} f\left(t\right) dt.$$

Notice the accessory attached to the integral symbol, the tiny $t = 0$ and the tiny x. If you remove the accessory, then it is an <u>in</u>definite integral. But with that accessory, it is <u>definite</u> integral. Now, the meaning of this is as follows:

Definition. The <u>definite integral</u> of $f\left(t\right)$ over the interval $\left[0,\ x\right]$ is simply

$$\int_{t=0}^{x} f\left(t\right) dt \;=\; x \cdot M\left(f\right)\left(x\right).$$

If you apply this definition, then we immediately get

Formula. Let n be a positive integer; $n = 1, 2, 3, 4, \cdots$. Then

$$\int_{t=0}^{x} t^n \, dt \;=\; \frac{1}{n+1} x^{n+1}.$$

Formula.

$$\int_{t=0}^{x} \cos t \, dt \;=\; \sin x, \qquad \int_{t=0}^{x} \sin t \, dt \;=\; 1 - \cos x.$$

Example 1. $$\int_{t=0}^{1} t^2\ dt \;=\; \frac{1}{2+1}1^{2+1} \;=\; 3.$$

Example 2. $$\int_{t=0}^{2} t^4\ dt \;=\; \frac{1}{4+1}2^{4+1} \;=\; \frac{32}{5}.$$

Example 3. $$\int_{t=0}^{\frac{\pi}{2}} \cos t\ dt \;=\; \sin\frac{\pi}{2} \;=\; 1.$$

Example 4. $$\int_{t=0}^{\frac{\pi}{3}} \cos t\ dt \;=\; \sin\frac{\pi}{3} \;=\; \frac{\sqrt{3}}{2}.$$

Example 5. $$\int_{t=0}^{\frac{\pi}{2}} \sin t\ dt \;=\; 1 - \cos\frac{\pi}{2} \;=\; 1.$$

Example 6. $$\int_{t=0}^{\frac{\pi}{3}} \sin t\ dt \;=\; 1 - \cos\frac{\pi}{3} \;=\; \frac{1}{2}.$$

Exercise 1. Evaluate

(1) $\displaystyle\int_{t=0}^{4} t\ dt.$ (2) $\displaystyle\int_{t=0}^{7} t^3\ dt.$

(3) $\displaystyle\int_{t=0}^{\frac{\pi}{4}} \cos t\ dt.$ (4) $\displaystyle\int_{t=0}^{\frac{\pi}{6}} \cos t\ dt.$

(5) $\displaystyle\int_{t=0}^{\frac{\pi}{4}} \sin t\ dt.$ (6) $\displaystyle\int_{t=0}^{\frac{\pi}{6}} \sin t\ dt.$

[**Answers**]**:** (1) 8. (2) $\dfrac{2401}{4}$. (3) $\dfrac{1}{\sqrt{2}}$.

(4) $\dfrac{1}{2}$. (5) $1 - \dfrac{1}{\sqrt{2}}$. (6) $1 - \dfrac{\sqrt{3}}{2}$.

• We have just defined the definite integral over an interval whose left-end is 0, namely, an interval of the form $\left[0,\ x\right]$. This is too restrictive. It makes sense to define the notion of definite integrals over an arbitrary interval, namely,

$$\int_{t=y}^{x} f(t)\,dt,$$

where y is not necessarily 0. This is taken care of by the following:

Fundamental theorem.

Suppose an antiderivative of $f(t)$ is $F(t)$. Then

$$\boxed{\int_{t=y}^{x} f(t)\,dt \;=\; F(x) - F(y).}$$

Notation. It is convenient to write the above theorem as

$$\boxed{\int_{t=y}^{x} f(t)\,dt \;=\; \Big[\, F(t) \,\Big]_{t=y}^{x}.}$$

Here, $\Big[\, F(t) \,\Big]_{t=y}^{x}$ simply means $F(x) - F(y)$.

Let's use some examples to illustrate how the evaluation goes:

Example 7.

$$\int_{t=2}^{3} t^2\ dt \;=\; \left[\,\frac{1}{3}t^3\,\right]_{t=2}^{3}$$

$$= \frac{1}{3}3^3 - \frac{1}{3}2^3 = \frac{19}{3}.$$

Example 8.

$$\int_{t=-1}^{1} t^6 \; dt \;=\; \Big[\; \frac{1}{7} t^7 \;\Big]_{t=-1}^{1} \;=\; \frac{1}{7} 1^7 \;-\; \frac{1}{7} \Big(-1 \Big)^7 \;=\; \frac{2}{7}.$$

• The following is inferred by what we have worked out today and 'Fundamental Theorem' above.

Quick Facts.

(1) An antiderivative of $\boxed{\cos x}$ is $\boxed{\sin x}$.

(2) An antiderivative of $\boxed{\sin x}$ is $\boxed{-\cos x}$.

Example 9.

$$\int_{t=\frac{\pi}{6}}^{\frac{\pi}{2}} \cos t \; dt \;=\; \Big[\; \sin t \;\Big]_{t=\frac{\pi}{6}}^{\frac{\pi}{2}} \;=\; \sin \frac{\pi}{2} \;-\; \sin \frac{\pi}{6} \;=\; 1 - \frac{1}{2} \;=\; \frac{1}{2}.$$

Example 10.

$$\int_{t=-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin t \; dt \;=\; \Big[\; -\cos t \;\Big]_{t=-\frac{\pi}{4}}^{\frac{\pi}{2}} \;=\; \Big(-\cos \frac{\pi}{2} \Big) \;-\; \Big(-\cos \Big(-\frac{\pi}{4} \Big) \Big) \;=\; 0 - \Big(-\frac{1}{\sqrt{2}} \Big) \;=\; \frac{1}{\sqrt{2}}.$$

Exercise 2. Evaluate

(1) $$\int_{t=-2}^{1} t^2 \; dt.$$

(2) $$\int_{t=1}^{\frac{3}{2}} t^5 \; dt.$$

(3) $$\int_{t=\frac{\pi}{4}}^{\frac{\pi}{3}} \cos t \; dt.$$

(4) $$\int_{t=-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos t \; dt.$$

(5) $$\int_{t=\frac{\pi}{6}}^{\frac{\pi}{4}} \sin t \; dt.$$

(6) $$\int_{t=-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin t \; dt.$$

[**Answers**]: (1) 3. (2) $\dfrac{665}{384}$. (3) $\dfrac{\sqrt{3}-\sqrt{2}}{2}$.

(4) $\dfrac{3}{2}$. (5) $\dfrac{\sqrt{3}-\sqrt{2}}{2}$. (6) $\dfrac{1}{\sqrt{2}}$.