Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XXXIV

April 22 (Wed), 2015

Instructor: Yasuyuki Kachi

Line #: 52920.

§34. Mean values.

Question 1. What is the mean of

$$1 \qquad \text{and} \qquad 9?$$

The sum of these numbers is 10. So, the answer is 10 divided by 2, which is 5. To summarize:

$$\text{the mean of } 1 \text{ and } 9 \;=\; \frac{1+9}{2} \;=\; 5.$$

⋆ More generally:

$$\text{The mean of } a \text{ and } b \text{ is } \frac{a+b}{2}.$$

Question 2. What is the mean of

$$7, \qquad -1 \qquad \text{and} \qquad 12?$$

The sum of these numbers is 18. This time, there are three numbers involved. So, the answer is 18 divided by 3, which is 6. To summarize:

$$\text{the mean of } 7,\ -1 \text{ and } 12 \;=\; \frac{7+\left(-1\right)+12}{3} \;=\; 6.$$

⋆ More generally:

$$\text{The mean of } a,\ b \text{ and } c \text{ is } \frac{a+b+c}{3}.$$

When four or more numbers are involved, it is the same. Namely, agree:

(1) the mean of a_1 is $\dfrac{a_1}{1}$.

(2) the mean of a_1 and a_2 is $\dfrac{a_1 + a_2}{2}$.

(3) the mean of a_1, a_2 and a_3 is $\dfrac{a_1 + a_2 + a_3}{3}$.

(4) the mean of a_1, a_2, a_3 and a_4 is $\dfrac{a_1 + a_2 + a_3 + a_4}{4}$.

(5) the mean of a_1, a_2, a_3, a_4 and a_5 is $\dfrac{a_1 + a_2 + a_3 + a_4 + a_5}{5}$.

(6) the mean of a_1, a_2, a_3, a_4, a_5 and a_6 is $\dfrac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{6}$.

$\vdots$

Exercise 1. (1) Find the mean of 2.

(2) Find the mean of 3 and –7.

(3) Find the mean of 3, 5 and 17.

(4) Find the mean of 4, 6, 8 and 10.

(5) Find the mean of –1, 2, –3, 4 and –5.

(6) Find the mean of 96, 48, 24, 12, 6 and 3.

[**Answers**]: (1) 2. (2) –2. (3) $\dfrac{25}{3}$. (4) 7.

(5) $-\dfrac{3}{5}$. (6) $\dfrac{63}{2}$.

- In the above, for no compelling reason, let's

$$\begin{array}{lll} \text{rewrite} & a_1 & \text{as} \;\; f(1); \\ \text{rewrite} & a_2 & \text{as} \;\; f(2); \\ \text{rewrite} & a_3 & \text{as} \;\; f(3); \\ \text{rewrite} & a_4 & \text{as} \;\; f(4); \\ \text{rewrite} & a_5 & \text{as} \;\; f(5); \\ \text{rewrite} & a_6 & \text{as} \;\; f(6); \\ & \vdots & \quad \vdots \end{array}$$

Thus

(1) the mean of $f(1)$ is $\dfrac{f(1)}{1}$.

(2) the mean of $f(1)$ and $f(2)$ is $\dfrac{f(1)+f(2)}{2}$.

(3) the mean of $f(1)$, $f(2)$ and $f(3)$ is $\dfrac{f(1)+f(2)+f(3)}{3}$.

(4) the mean of $f(1)$, $f(2)$, $f(3)$ and $f(4)$ is $\dfrac{f(1)+f(2)+f(3)+f(4)}{4}$.

(5) the mean of $f(1)$, $f(2)$, $f(3)$, $f(4)$ and $f(5)$ is $\dfrac{f(1)+f(2)+f(3)+f(4)+f(5)}{5}$.

(6) the mean of $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$ and $f(6)$ is

$$\frac{f(1)+f(2)+f(3)+f(4)+f(5)+f(6)}{6}.$$

$\vdots$

Example 1. Let $f(x) = x^2$. Then

$$\begin{aligned} f(1) &= 1^2, \\ f(2) &= 2^2, \\ f(3) &= 3^2, \\ f(4) &= 4^2, \\ f(5) &= 5^2, \\ f(6) &= 6^2, \\ &\vdots \end{aligned}$$

Accordingly

(1) the mean of $f(1)$ is $\frac{1^2}{1}$.

(2) the mean of $f(1)$ and $f(2)$ is $\frac{1^2+2^2}{2}$.

(3) the mean of $f(1)$, $f(2)$ and $f(3)$ is $\frac{1^2+2^2+3^2}{3}$.

(4) the mean of $f(1)$, $f(2)$, $f(3)$ and $f(4)$ is $\frac{1^2+2^2+3^2+4^2}{4}$.

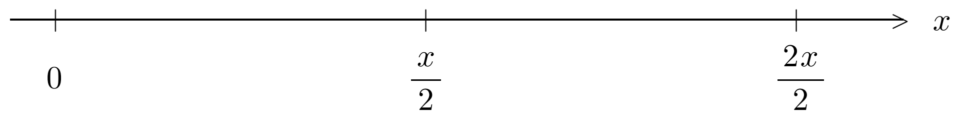
(5) the mean of $f(1)$, $f(2)$, $f(3)$, $f(4)$ and $f(5)$ is $\frac{1^2+2^2+3^2+4^2+5^2}{5}$.

(6) the mean of $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$ and $f(6)$ is $\frac{1^2+2^2+3^2+4^2+5^2+6^2}{6}$.

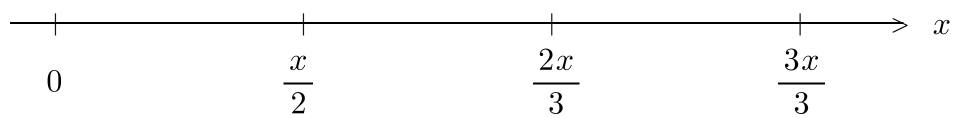
$\vdots$

• In a similar vein, suppose $f(x)$ is given. Somehow the following kind of means play important roles in the subsequent discussion:

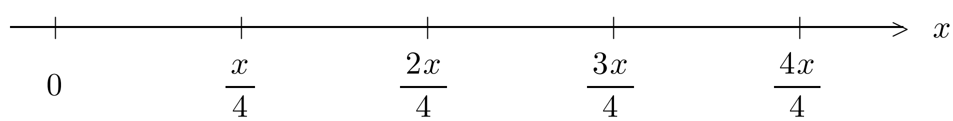
∘ the mean of $f\left(\frac{x}{2}\right)$ and $f\left(\frac{2x}{2}\right)$.



∘ the mean of $f\left(\frac{x}{3}\right)$, $f\left(\frac{2x}{3}\right)$ and $f\left(\frac{3x}{3}\right)$.



∘ the mean of $f\left(\frac{x}{4}\right)$, $f\left(\frac{2x}{4}\right)$, $f\left(\frac{3x}{4}\right)$ and $f\left(\frac{4x}{4}\right)$.



⋮

Example 2. Once again, let $f(x) = x^2$. Then

$$f\left(\frac{x}{2}\right)=\left(\frac{x}{2}\right)^2,\quad f\left(\frac{2x}{2}\right)=\left(\frac{2x}{2}\right)^2,$$

$$f\left(\frac{x}{3}\right)=\left(\frac{x}{3}\right)^2,\quad f\left(\frac{2x}{3}\right)=\left(\frac{2x}{3}\right)^2,\quad f\left(\frac{3x}{3}\right)=\left(\frac{3x}{3}\right)^2,$$

$$f\left(\frac{x}{4}\right)=\left(\frac{x}{4}\right)^2,\quad f\left(\frac{2x}{4}\right)=\left(\frac{2x}{4}\right)^2,\quad f\left(\frac{3x}{4}\right)=\left(\frac{3x}{4}\right)^2,\quad f\left(\frac{4x}{4}\right)=\left(\frac{4x}{4}\right)^2,$$

⋮ ⋮ ⋮

Accordingly

- the mean of $f\left(\dfrac{x}{2}\right)$ and $f\left(\dfrac{2x}{2}\right)$ is $\dfrac{\left(\dfrac{x}{2}\right)^2+\left(\dfrac{2x}{2}\right)^2}{2}$,
- the mean of $f\left(\dfrac{x}{3}\right)$, $f\left(\dfrac{2x}{3}\right)$ and $f\left(\dfrac{3x}{3}\right)$ is $\dfrac{\left(\dfrac{x}{3}\right)^2+\left(\dfrac{2x}{3}\right)^2+\left(\dfrac{3x}{3}\right)^2}{3}$,
- the mean of $f\left(\dfrac{x}{4}\right)$, $f\left(\dfrac{2x}{4}\right)$, $f\left(\dfrac{3x}{4}\right)$ and $f\left(\dfrac{4x}{4}\right)$ is

$$\frac{\left(\dfrac{x}{4}\right)^2+\left(\dfrac{2x}{4}\right)^2+\left(\dfrac{3x}{4}\right)^2+\left(\dfrac{4x}{4}\right)^2}{4},$$

$$\vdots$$

These are clearly rewritten as

$$\left(1^2+2^2\right)\frac{x^2}{2^3},$$

$$\left(1^2+2^2+3^2\right)\frac{x^2}{3^3},$$

$$\left(1^2+2^2+3^2+4^2\right)\frac{x^2}{4^3},$$

$$\vdots$$

As you extrapolate the patterns, you agree that the following is true:

- the mean of $f\left(\dfrac{x}{n}\right)$, $f\left(\dfrac{2x}{n}\right)$, $f\left(\dfrac{3x}{n}\right)$, $\cdots$ $f\left(\dfrac{nx}{n}\right)$ is

$$\left(\underbrace{1^2+2^2+3^2+\cdots+n^2}\right)\frac{x^2}{n^3}.$$

As for the underlined part, let's recall the formula:

Formula. $\big($from "Review of Lectures – XXVII"$\big)$.

$$\boxed{1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n}.$$

So the above is paraphrased as follows:

- the mean of $f\Big(\dfrac{x}{n}\Big)$, $f\Big(\dfrac{2x}{n}\Big)$, $f\Big(\dfrac{3x}{n}\Big)$, $\cdots$, $f\Big(\dfrac{nx}{n}\Big)$ is

$$\Big(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\Big)\frac{x^2}{n^3},$$

that is,

$$\left(\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{n} + \frac{1}{6}\cdot\frac{1}{n^2}\right)x^2.$$

It is useful for later purpose to examine the state of this quantity when n grows arbitrarily large. When n grows, like

$$n = 1000,$$
$$n = 1000000,$$
$$n = 1000000000,$$
$$\vdots$$

then accordingly $\dfrac{1}{n}$ and $\dfrac{1}{n^2}$ become negligible:

$$\begin{array}{lll} n = 1000 & \Longrightarrow \dfrac{1}{n} = 0.001, & \dfrac{1}{n^2} = 0.000001, \\ n = 1000000 & \Longrightarrow \dfrac{1}{n} = 0.000001, & \dfrac{1}{n^2} = 0.000000000001, \\ n = 1000000000 & \Longrightarrow \dfrac{1}{n} = 0.000000001, & \dfrac{1}{n^2} = 0.000000000000000001, \\ & \vdots & \end{array}$$

More precisely,

$$\lim_{n \to \infty} \frac{1}{n} = 0, \qquad \lim_{n \to \infty} \frac{1}{n^2} = 0.$$

Hence

$$\lim_{n \to \infty} \left(\frac{1}{3} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{6} \cdot \frac{1}{n^2} \right) x^2 = \frac{1}{3} x^2.$$

- **Summary of Example 2.**

" *For*

$$f(x) = x^2,$$

the mean of $f\left(\frac{x}{n}\right),\ f\left(\frac{2x}{n}\right),\ f\left(\frac{3x}{n}\right),\ \cdots,\ f\left(\frac{nx}{n}\right),$ *as* $n \longrightarrow \infty,$

is

$$\frac{1}{3} x^2. \text{''}$$

Example 3. Let $f(x) = x^3$. Once again, let's consider

- the mean of $f\left(\frac{x}{2}\right)$ and $f\left(\frac{2x}{2}\right)$,
- the mean of $f\left(\frac{x}{3}\right)$, $f\left(\frac{2x}{3}\right)$ and $f\left(\frac{3x}{3}\right)$,
- the mean of $f\left(\frac{x}{4}\right)$, $f\left(\frac{2x}{4}\right)$, $f\left(\frac{3x}{4}\right)$ and $f\left(\frac{4x}{4}\right)$,

  ⋮

- the mean of $f\left(\frac{x}{n}\right)$, $f\left(\frac{2x}{n}\right)$, $f\left(\frac{3x}{n}\right)$ $\cdots$, $f\left(\frac{nx}{n}\right)$.

The answers are as follows:

$$\left(1^3 + 2^3\right)\frac{x^3}{2^4},$$

$$\left(1^3 + 2^3 + 3^3\right)\frac{x^3}{3^4},$$

$$\left(1^3 + 2^3 + 3^3 + 4^3\right)\frac{x^3}{4^4},$$

$$\vdots$$

$$\underbrace{\left(1^3 + 2^3 + 3^3 + \cdots + n^3\right)}\frac{x^3}{n^4}.$$

Once again, as for the underlined part, recall the formula:

Formula. (from "Review of Lectures – XXVII")**.**

$$\boxed{1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2},$$

We may accordingly rewrite the last one as

$$\left(\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2\right)\frac{x^3}{n^4},$$

which equals

$$\left(\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{n} + \frac{1}{4}\cdot\frac{1}{n^2}\right)x^3.$$

Now, limit-wise

$$\lim_{n\to\infty}\left(\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{n} + \frac{1}{4}\cdot\frac{1}{n^2}\right)x^3 = \frac{1}{4}x^3.$$

In sum:

- **Summary of Example 3.**

“ *For*

$$f\left(x\right) \;=\; x^3,$$

the mean of $\quad f\Big(\frac{x}{n}\Big),\ f\Big(\frac{2x}{n}\Big),\ f\Big(\frac{3x}{n}\Big),\ \cdots,\ f\left(\frac{nx}{n}\right),\quad$ *as* $\quad n \longrightarrow \infty,$

is

$$\frac{1}{4}x^3. \text{ ”}$$

Now, nothing stops us from considering the same problem for

$$f\left(x\right) \;=\; x^4,$$

$$f\left(x\right) \;=\; x^5,$$

$$f\left(x\right) \;=\; x^6,$$

$$\vdots$$

By extrapolation, we conclude:

the mean of $\quad f\Big(\frac{x}{n}\Big),\ f\Big(\frac{2x}{n}\Big),\ f\Big(\frac{3x}{n}\Big),\ \cdots,\ f\left(\frac{nx}{n}\right),\quad$ as $\quad n \longrightarrow \infty,\quad$ is

$$\frac{1}{5}x^4, \qquad \text{when} \qquad f\left(x\right) \;=\; x^4,$$

$$\frac{1}{6}x^5, \qquad \text{when} \qquad f\left(x\right) \;=\; x^5,$$

$$\frac{1}{7}x^6, \qquad \text{when} \qquad f\left(x\right) \;=\; x^6,$$

$$\vdots$$

• If you utilize the formula in "Review of Lectures – XXVII", page 1, then you will be able to pull more precise results (below). Before presenting them, we introduce the following notation, which is convenient.

Notation. For a given $f(x)$, the notation $M_n(f)(x)$ stands for the mean of

$$f\left(\frac{x}{n}\right),\ f\left(\frac{2x}{n}\right),\ f\left(\frac{3x}{n}\right),\ \cdots,\ f\left(\frac{nx}{n}\right).$$

Also, the notation $M(f)(x)$ stands for the limit

$$\lim_{n \to \infty} M_n(f)(x).$$

Results.

(1) $f(x) = x \implies M_n(f)(x) = \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{n}\right) x,$

$$M(f)(x) = \frac{1}{2} x.$$

(2) $f(x) = x^2 \implies M_n(f)(x) = \left(\frac{1}{3} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{6} \cdot \frac{1}{n^2}\right) x^2,$

$$M(f)(x) = \frac{1}{3} x^2.$$

(3) $f(x) = x^3 \implies M_n(f)(x) = \left(\frac{1}{4} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{4} \cdot \frac{1}{n^2}\right) x^3,$

$$M(f)(x) = \frac{1}{4} x^3.$$

(4) $f(x) = x^4 \implies M_n(f)(x) = \left(\frac{1}{5} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{3} \cdot \frac{1}{n^2} - \frac{1}{30} \cdot \frac{1}{n^4} \right) x^4,$

$$M(f)(x) = \frac{1}{5} x^4.$$

(5) $f(x) = x^5 \implies M_n(f)(x) = \left(\frac{1}{6} + \frac{1}{2} \cdot \frac{1}{n} + \frac{5}{12} \cdot \frac{1}{n^2} - \frac{1}{12} \cdot \frac{1}{n^4} \right) x^5,$

$$M(f)(x) = \frac{1}{6} x^5.$$

Exercise 2. Find $M_n(f)(x)$ and $M(f)(x)$ for

(6) $f(x) = x^6.$

Use the following, if necessary.

Formula.

$$\boxed{1^6 + 2^6 + 3^6 + \cdots + n^6 = \frac{1}{7} n^7 + \frac{1}{2} n^6 + \frac{1}{2} n^5 - \frac{1}{6} n^3 + \frac{1}{42} n}.$$

[**Answer**]:

$$M_n(f)(x) = \left(\frac{1}{7} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{2} \cdot \frac{1}{n^2} - \frac{1}{6} \cdot \frac{1}{n^4} + \frac{1}{42} \cdot \frac{1}{n^6} \right) x^6,$$

$$M(f)(x) = \frac{1}{7} x^6.$$