Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XXXIII

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§33. Trigonometry – III.

In the previous lecture we calculated the distance between

$$P = \left(\cos\frac{\pi}{4},\ \sin\frac{\pi}{4}\right), \qquad Q = \left(\cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right).$$

Now, we can more generally consider the following question:

Question 1. Find the distance between

$$P = \Big(\cos\theta,\ \sin\theta\Big), \qquad Q = \Big(\cos\phi,\ \sin\phi\Big).$$

where θ and ϕ are arbitrary.

[**Solution**]:

$$\left|PQ\right|$$

$$= \sqrt{\Big((\cos\theta) - (\cos\phi)\Big)^2 + \Big((\sin\theta) - (\sin\phi)\Big)^2}$$

$$= \sqrt{\Big(\cos\theta\Big)^2 - 2\Big(\cos\theta\Big)\Big(\cos\phi\Big) + \Big(\cos\phi\Big)^2 + \Big(\sin\theta\Big)^2 - 2\Big(\sin\theta\Big)\Big(\sin\phi\Big) + \Big(\sin\phi\Big)^2}$$

$$= \sqrt{1 + 1 - 2\Big(\cos\theta\Big)\Big(\cos\phi\Big) - 2\Big(\sin\theta\Big)\Big(\sin\phi\Big)}$$

$$= \sqrt{2 - 2\Big(\cos\theta\Big)\Big(\cos\phi\Big) - 2\Big(\sin\theta\Big)\Big(\sin\phi\Big)}\ .$$

Now, exactly as we previously did in the case $\theta = \dfrac{\pi}{4}$ and $\phi = \dfrac{\pi}{6}$, we pose:

Question 2. Find the distance between

$$R = \Big(\cos\big(\theta - \phi\big),\ \sin\big(\theta - \phi\big)\Big), \qquad S = \big(1,\ 0\big).$$

Explain why $\big|PQ\big|$ and $\big|RS\big|$ are equal. Then use that fact and the result of Question 1 above to write up $\cos\big(\theta - \phi\big)$ in terms of $\cos\theta$, $\sin\theta$, $\cos\phi$ and $\sin\phi$.

[**Solution**]:

P and Q are both lying in the unit circle such that the angle $\angle POQ$ equals $\theta - \phi$. R and S are also both lying in the unit circle such that the angle $\angle ROS$ equals $\theta - \phi$. Hence $\big|PQ\big|$ and $\big|RS\big|$ are naturally equal. Now,

$$\begin{aligned}
\big|RS\big| &= \sqrt{\Big(\big(\cos(\theta - \phi)\big) - 1\Big)^2 + \Big(\sin(\theta - \phi)\Big)^2} \\
&= \sqrt{\Big(\cos(\theta - \phi)\Big)^2 - 2\Big(\cos(\theta - \phi)\Big) + 1 + \Big(\sin(\theta - \phi)\Big)^2} \\
&= \sqrt{1 - 2\Big(\cos(\theta - \phi)\Big) + 1} \\
&= \sqrt{2 - 2\Big(\cos(\theta - \phi)\Big)}\,.
\end{aligned}$$

This equals

$$\big|PQ\big| = \sqrt{2 - 2\Big(\cos\theta\Big)\Big(\cos\phi\Big) - 2\Big(\sin\theta\Big)\Big(\sin\phi\Big)}\,.$$

So

$$2 - 2\left(\cos\left(\theta - \phi\right)\right) = 2 - 2\left(\cos\theta\right)\left(\cos\phi\right) - 2\left(\sin\theta\right)\left(\sin\phi\right).$$

Solve this for $\cos\left(\theta - \phi\right)$:

$$\cos\left(\theta - \phi\right) = \left(\cos\theta\right)\left(\cos\phi\right) + \left(\sin\theta\right)\left(\sin\phi\right).$$

⋆ Let's highlight this result:

Axiom 1. $\boxed{\cos\left(\theta - \phi\right) = \left(\cos\theta\right)\left(\cos\phi\right) + \left(\sin\theta\right)\left(\sin\phi\right)}$.

• Now, we shoot for pulling an analogous identity for $\sin\left(\theta - \phi\right)$. Toward that goal, working on the following two questions (Question 1′, Question 2′) is the correct step to take:

Question 1′. Find the distance between

$$P' = \left(\sin\theta,\ \cos\theta\right), \qquad Q' = \left(\cos\phi,\ -\sin\phi\right).$$

where θ and ϕ are arbitrary.

The answer is,

$$\left|\,P'Q'\,\right| = \sqrt{2 - 2\left(\sin\theta\right)\left(\cos\phi\right) + 2\left(\cos\theta\right)\left(\sin\phi\right)}\,.$$

The calculation is similar to Question 1.

Question 2′. Find the distance between

$$R' \;=\; \Big(\sin\big(\theta-\phi\big),\ \ \cos\big(\theta-\phi\big)\Big), \qquad\qquad S \;=\; \big(1,\ 0\big).$$

Explain why $\big|\,P'Q'\,\big|$ and $\big|\,R'S\,\big|$ are equal. Then use that fact and the result of Question 1 above to write up $\sin\big(\theta-\phi\big)$ in terms of $\cos\theta$, $\sin\theta$, $\cos\phi$ and $\sin\phi$.

[**Solution**]**:** You can solve this exactly the same way as Question 2:

P' and Q' are both lying in the unit circle such that the angle $\angle P'OQ'$ equals $\dfrac{\pi}{2}-\theta+\phi$. R and S are also both lying in the unit circle such that the angle $\angle R'OS$ equals $\dfrac{\pi}{2}-\theta+\phi$. Hence $\big|\,P'Q'\,\big|$ and $\big|\,R'S\,\big|$ are naturally equal. Now,

$$\big|\,R'S\,\big| \;=\; \sqrt{2\;-\;2\Big(\sin\,\big(\theta-\phi\big)\Big)}\,.$$

The calculation is similar to the calculation of $\big|\,RS\,\big|$. This equals

$$\big|\,P'Q'\,\big| \;=\; \sqrt{2\;-\;2\Big(\sin\,\theta\Big)\Big(\cos\,\phi\Big)\;+\;2\Big(\cos\,\theta\Big)\Big(\sin\,\phi\Big)}\,.$$

So

$$2\;-\;2\Big(\sin\,\big(\theta-\phi\big)\Big) \;=\; 2\;-\;2\Big(\sin\theta\Big)\Big(\cos\phi\Big)\;+\;2\Big(\cos\theta\Big)\Big(\sin\phi\Big).$$

Solve this for $\sin\,\big(\theta-\phi\big)$:

$$\sin\,\big(\theta-\phi\big) \;=\; \Big(\sin\theta\Big)\Big(\cos\phi\Big)\;-\;\Big(\cos\theta\Big)\Big(\sin\phi\Big).$$

⋆ Let's pair 'Axiom 1' which we deduced earlier with the above new result, and highlight them together:

Axiom 1.	$\cos\Big(\theta - \phi\Big) \;=\; \Big(\cos\theta\Big)\Big(\cos\phi\Big) + \Big(\sin\theta\Big)\Big(\sin\phi\Big)$.
Axiom 2.	$\sin\Big(\theta - \phi\Big) \;=\; \Big(\sin\theta\Big)\Big(\cos\phi\Big) - \Big(\cos\theta\Big)\Big(\sin\phi\Big)$.

- How about analogs for $\cos\Big(\theta + \phi\Big)$ and for $\sin\Big(\theta + \phi\Big)$?

For that matter, we rely on Axiom 1 and Axiom 2, and proceed as follows: First, set $\theta - \phi = \alpha$. Thus $\theta = \alpha + \phi$. Then the above two identities are

$$\Big(\cos\phi\Big)\Big(\cos(\alpha + \phi)\Big) + \Big(\sin\phi\Big)\Big(\sin(\alpha + \phi)\Big) \;=\; \cos\alpha,$$

$$-\Big(\sin\phi\Big)\Big(\cos(\alpha + \phi)\Big) + \Big(\cos\phi\Big)\Big(\sin(\alpha + \phi)\Big) \;=\; \sin\alpha.$$

These two equations are regarded as

$$\begin{cases} ax + by = p, \\ cx + dy = q, \end{cases}$$

where

$$a = \cos\phi, \qquad b = \sin\phi, \qquad p = \cos\alpha,$$

$$c = -\sin\phi, \qquad d = \cos\phi, \qquad q = \sin\alpha,$$

$$x = \cos\Big(\theta + \phi\Big), \qquad \text{and} \qquad y = \sin\Big(\theta + \phi\Big).$$

Clearly $ad - bc = 1$. Thus we rely on the result in "Supplement":

$$\boxed{x = dp - bq, \qquad y = -cp + aq}\;.$$

That is:

$$\cos\Big(\alpha+\phi\Big) = \Big(\cos\phi\Big)\Big(\cos\alpha\Big) - \Big(\sin\phi\Big)\Big(\sin\alpha\Big),$$

$$\sin\Big(\alpha+\phi\Big) = \Big(\sin\phi\Big)\Big(\cos\alpha\Big) + \Big(\cos\phi\Big)\Big(\sin\alpha\Big).$$

At this point, we are free to replace the letter α with θ. Let's go ahead and make that replacement, and highlight:

Axiom 3. $$\boxed{\cos\Big(\theta+\phi\Big) = \Big(\cos\theta\Big)\Big(\cos\phi\Big) - \Big(\sin\theta\Big)\Big(\sin\phi\Big)}.$$

Axiom 4. $$\boxed{\sin\Big(\theta+\phi\Big) = \Big(\sin\theta\Big)\Big(\cos\phi\Big) + \Big(\cos\theta\Big)\Big(\sin\phi\Big)}.$$

⋆ In Axiom 1 and Axiom 2, set $\phi = \theta$ and we get

Double angle formula for cos.

$$\boxed{\cos\Big(2\theta\Big) = \Big(\cos\theta\Big)^2 - \Big(\sin\theta\Big)^2}.$$

Double angle formula for cos – version 2.

$$\boxed{\cos\Big(2\theta\Big) = 2\Big(\cos\theta\Big)^2 - 1}.$$

Double angle formula for cos – version 3.

$$\boxed{\cos\Big(2\theta\Big) = 1 - 2\Big(\sin\theta\Big)^2}.$$

Double angle formula for sin.

$$\boxed{\sin\Big(2\theta\Big) = 2\Big(\cos\theta\Big)\Big(\sin\theta\Big)}.$$

⋆ From Axiom 1 and Axiom 3, we have

$$\textbf{Formula A.} \quad \boxed{2\Big(\cos\,\theta\Big)\Big(\cos\,\phi\Big) \;=\; \cos\,\Big(\theta-\phi\Big) + \cos\,\Big(\theta+\phi\Big)},$$

$$\textbf{Formula B.} \quad \boxed{2\Big(\sin\,\theta\Big)\Big(\sin\,\phi\Big) \;=\; \cos\,\Big(\theta-\phi\Big) - \cos\,\Big(\theta+\phi\Big)}.$$

⋆ Similarly, from Axiom 2 and Axiom 4, we have

$$\textbf{Formula C.} \quad \boxed{2\Big(\sin\,\theta\Big)\Big(\cos\,\phi\Big) \;=\; \sin\,\Big(\theta-\phi\Big) + \sin\,\Big(\theta+\phi\Big)}.$$

- Today's final topic is

$$\boxed{\lim_{\theta\to 0} \frac{\sin\,\theta}{\theta}} \quad .$$

This is somehow very important. Don't ask me yet how so. As for this, consider

$$P \;=\; \Big(\cos\,\theta,\;\sin\,\theta\Big), \qquad\qquad P_0 \;=\; \Big(1,\;0\Big),$$

$$T \;=\; \left(1,\;\frac{\sin\,\theta}{\cos\,\theta}\right).$$

Assume $\theta > 0$. T is the unique point lying in the extension of the segment OP whose x-coordinate reading is 1. By drawing the figure, we have

$$\Big(\text{the area of } \triangle POP_0\Big) \;<\; \Big(\text{the area of circle sector } POP_0\Big)$$
$$<\; \Big(\text{the area of } \triangle TOP_0\Big).$$

This translates into

$$\frac{\sin\,\theta}{2} \underset{(*)}{<} \frac{\theta}{2} \underset{(\#)}{<} \frac{\sin\,\theta}{2\,\cos\,\theta}.$$

Here, we use the fact that the area of the entire disc of radius 1 equals π $\Big($the middle term is $\dfrac{\theta}{2\pi}$ times the area of the entire unit disc$\Big)$. From $(*)$ we have

$$\frac{\sin\,\theta}{\theta} < 1,$$

whereas from $(\#)$ we have

$$\cos\,\theta < \frac{\sin\,\theta}{\theta}.$$

In short,

$$\cos\,\theta < \frac{\sin\,\theta}{\theta} < 1.$$

From this it is inferred that

$$\lim_{\theta \longrightarrow 0} \frac{\sin\,\theta}{\theta} = 1.$$

Indeed,

$$\lim_{\theta \longrightarrow 0} \cos\,\theta = \cos\,0 = 1.$$

- To highlight:

Formula D.

$$\boxed{\lim_{\theta \longrightarrow 0} \frac{\sin\,\theta}{\theta} = 1.}\ .$$

$\star$ Technically, this last implication is due to the so-called "squeeze theorem". The squeeze theorem asserts that, if

$$f(x) \;<\; g(x) \;<\; h(x),$$

for $a < x < b$ (where a is some negative number, a is some positive number), and moreover if

$$\lim_{x \longrightarrow 0} f(x) \;=\; \lim_{x \longrightarrow 0} h(x),$$

then these limits further equal $\lim_{x \longrightarrow 0} g(x)$.