Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XXXII

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§32. Distance.

Today we talk about the distance — the distance between two points lying in the xy-coordiate system, to be precise.

Definition. Let P and Q be two points, both lying in the xy-coordiate system. Suppose the coordinate reading of P and Q are given by

$$P = \Big(a,\ b\Big), \qquad \text{and} \qquad Q = \Big(c,\ d\Big),$$

respectively. Then the distance $\big|PQ\big|$ between P and Q is

$$\boxed{\sqrt{\Big(c - a\Big)^2 + \Big(d - b\Big)^2}} \quad .$$

⋆ The following is a speial case:

Let P be a point, lying in the xy-coordiate system. Suppose the coordinate reading of P is given by

$$P = \Big(a,\ b\Big).$$

Meanwhile, let O be the coordinate origin:

$$O = \Big(0,\ 0\Big).$$

Then the distance $\big|PO\big|$ between P and O is

$$\boxed{\sqrt{a^2 + b^2}} \quad .$$

Example 1. Let's find the distance between

$$P = \Big(3,\ 5\Big), \qquad Q = \Big(4,\ 7\Big).$$

By definition,

$$\begin{aligned} |\,PQ\,| &= \sqrt{\Big(4 - 3\Big)^2 + \Big(7 - 5\Big)^2} \\ &= \sqrt{1^2 + 2^2} \\ &= \sqrt{5}\,. \end{aligned}$$

Example 2. Let's find the distance between

$$P = \Big(1,\ -3\Big), \qquad Q = \Big(-2,\ 4\Big).$$

By definition,

$$\begin{aligned} |\,PQ\,| &= \sqrt{\Big(-2 - 1\Big)^2 + \Big(4 - (-3)\Big)^2} \\ &= \sqrt{(-3)^2 + 7^2} \\ &= \sqrt{9 + 49} \\ &= \sqrt{58}\,. \end{aligned}$$

Example 3. Let's find the distance between

$$P = \Big(-5,\ 0\Big), \qquad Q = \Big(6,\ 0\Big).$$

By definition,

$$\begin{aligned} |\,PQ\,| &= \sqrt{\Big(6 - (-5)\Big)^2 + \Big(0 - 0\Big)^2} \\ &= \sqrt{11^2 + 0^2} \\ &= \sqrt{11^2} \\ &= 11. \end{aligned}$$

Example 4. Let's find the distance between

$$P = \Big(12,\ 1\Big), \qquad O = \Big(0,\ 0\Big).$$

By definition,

$$\begin{aligned} |\,PO\,| &= \sqrt{12^2 + 1^2} \\ &= \sqrt{144 + 1} \\ &= \sqrt{145}\,. \end{aligned}$$

Exercise 1. Find the distance between P and Q.

(1) $P = \Big(2,\ 0\Big), \qquad Q = \Big(3,\ 1\Big).$

(2) $P = \Big(-4,\ 3\Big), \qquad Q = \Big(-1,\ 7\Big).$

(3) $P = \Big(-3,\ 4\Big), \qquad Q = \Big(2,\ 16\Big).$

(4) $P = \Big(100,\ 0\Big), \qquad Q = \Big(100,\ 1\Big).$

(5) $P = \Big(4,\ 1\Big), \qquad O = \Big(0,\ 0\Big).$

(6) $O = \Big(0,\ 0\Big), \qquad Q = \Big(8,\ 15\Big).$

[**Answers**]**:**

(1) $\sqrt{2}$. (2) 5. (3) 13. (4) 1.

(5) $\sqrt{17}$. (6) 17.

- **Most basic property of sin and cos.**

No matter what you do, please remember the following, which is extremely important:

$$\boxed{\Big(\cos\theta\Big)^2 + \Big(\sin\theta\Big)^2 = 1}\ .$$

This is true no matter what θ is.

Example 5. $$\left(\cos\frac{\pi}{9}\right)^2 + \left(\sin\frac{\pi}{9}\right)^2 = 1.$$

Example 6. $$\left(\cos\frac{\pi}{5}\right)^2 + \left(\sin\frac{\pi}{5}\right)^2 = 1.$$

Example 7. $$\left(\cos\frac{3\pi}{7}\right)^2 + \left(\sin\frac{3\pi}{7}\right)^2 = 1.$$

Example 8. $$\left(\cos\frac{\pi}{\sqrt{2}}\right)^2 + \left(\sin\frac{\pi}{\sqrt{2}}\right)^2 = 1.$$

Example 9. $$\left(\cos\left(-\frac{\pi}{15}\right)\right)^2 + \left(\sin\left(-\frac{\pi}{15}\right)\right)^2 = 1.$$

⋆ Can you paraphrase the above identity in terms of the distance? I bet you can.

“ The distance between

$$P = \left(\cos\theta,\ \sin\theta\right)$$

and the coordinate origin $O = \left(0,\ 0\right)$ is always 1. ”

⋆ Here is a further paraphrase:

“ The point

$$P = \left(\cos\theta,\ \sin\theta\right)$$

always lies in *the unit circle*, the circle with radius 1 centered at the origin.”

Here is one important exercise:

Exercise 2. Find the distance between

$$P = \left(\cos\frac{\pi}{4},\ \sin\frac{\pi}{4}\right), \qquad Q = \left(\cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right).$$

Solution: First, recall

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \qquad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2},$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \qquad \sin\frac{\pi}{6} = \frac{1}{2}.$$

Thus, by definition,

$$\begin{aligned}
|PQ| &= \sqrt{\left(\left(\cos\frac{\pi}{4}\right) - \left(\cos\frac{\pi}{6}\right)\right)^2 + \left(\left(\sin\frac{\pi}{4}\right) - \left(\sin\frac{\pi}{6}\right)\right)^2} \\
&= \sqrt{\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)^2} \\
&= \sqrt{\left(\frac{\sqrt{3} - \sqrt{2}}{2}\right)^2 + \left(\frac{1 - \sqrt{2}}{2}\right)^2} \\
&= \sqrt{\frac{3 + 2 - 2\sqrt{3}\cdot\sqrt{2}}{4} + \frac{1 + 2 - 2\cdot 1\cdot\sqrt{2}}{4}} \\
&= \sqrt{\frac{5 - 2\sqrt{6}}{4} + \frac{3 - 2\sqrt{2}}{4}} \\
&= \sqrt{\frac{5 - 2\sqrt{6} + 3 - 2\sqrt{2}}{4}} \\
&= \sqrt{\frac{8 - 2\sqrt{6} - 2\sqrt{2}}{4}} \quad \left(= \frac{1}{2}\sqrt{8 - 2\sqrt{6} - 2\sqrt{2}}\right).
\end{aligned}$$

Exercise 3. Let P and Q be as in Exercise 2. Let

$$R = \left(\cos\frac{\pi}{12},\ \sin\frac{\pi}{12}\right), \qquad S = \big(1,\ 0\big).$$

Explain why $\left|PQ\right|$ and $\left|RS\right|$ are equal. Then use that fact and the result of Exercise 2 to evaluate $\cos\frac{\pi}{12}$.

$\left[\underline{\textbf{Solution}}\right]$**:**

P and Q are both lying in the unit circle such that the angle $\angle POQ$ equals $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$. R and S are also both lying in the unit circle such that the angle $\angle ROS$ equals $\frac{\pi}{12} - 0 = \frac{\pi}{12}$. Hence $\left|PQ\right|$ and $\left|RS\right|$ are naturally equal. Now,

$$\begin{aligned}
\left|RS\right| &= \sqrt{\left(\left(\cos\frac{\pi}{12}\right) - 1\right)^2 + \left(\sin\frac{\pi}{12}\right)^2} \\
&= \sqrt{\left(\cos\frac{\pi}{12}\right)^2 - 2\left(\cos\frac{\pi}{12}\right) + 1 + \left(\sin\frac{\pi}{12}\right)^2} \\
&= \sqrt{1 - 2\left(\cos\frac{\pi}{12}\right) + 1} \\
&= \sqrt{2 - 2\left(\cos\frac{\pi}{12}\right)}\ .
\end{aligned}$$

This equals

$$\left|RS\right| = \sqrt{\frac{8 - 2\sqrt{6} - 2\sqrt{2}}{4}}\ .$$

So

$$2 - 2\left(\cos\frac{\pi}{12}\right) = \frac{8 - 2\sqrt{6} - 2\sqrt{2}}{4}.$$

Solve this for $\cos\dfrac{\pi}{12}$:

$$2\cos\frac{\pi}{12} = \frac{2\sqrt{6} + 2\sqrt{2}}{4}$$

$$\Longrightarrow \qquad \cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}.$$