Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – III

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§3. DOUBLE SUM OF CONSECUTIVE INTEGERS.

Okay. We have worked out

$$
\begin{aligned}
&1 = 1,\\
&1 + 2 = 3,\\
&1 + 2 + 3 = 6,\\
&1 + 2 + 3 + 4 = 10,\\
&1 + 2 + 3 + 4 + 5 = 15,\\
&1 + 2 + 3 + 4 + 5 + 6 = 21,\\
&1 + 2 + 3 + 4 + 5 + 6 + 7 = 28,\\
&1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36,\\
&1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45,\\
&1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55,\\
&\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \ddots
\end{aligned}
$$

Let me just highlight the answers:

$$1, \quad 3, \quad 6, \quad 10, \quad 15, \quad 21, \quad 28, \quad 36, \quad 45, \quad 55, \quad \cdots .$$

Now, by any chance, after the last class, have you thought about what if we just added up these numbers, as in

(1) $1 \ =?$
(2) $1 + 3 \ =?$
(3) $1 + 3 + 6 \ =?$
(4) $1 + 3 + 6 + 10 \ =?$
(5) $1 + 3 + 6 + 10 + 15 \ =?$
(6) $1 + 3 + 6 + 10 + 15 + 21 \ =?$
(7) $1 + 3 + 6 + 10 + 15 + 21 + 28 \ =?$
(8) $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 \ =?$
(9) $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 \ =?$
(10) $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 \ =?$

Don't ask me what for. Not yet. We can do these brute-force. But I get lazy. Is there any trick? Yes there is. But you have to rely on the power of algebra . Last time we tested the water first. This time around, let's do part (8) right off the bat. But before we begin our work, it is good to observe the following:

[**Clue**] We are supposed to do $1+3+6+10+15+21+28+36$, but

$$
\begin{array}{rcl}
1 & = & 1 \\
3 & = & 1+2 \\
6 & = & 1+2+3 \\
10 & = & 1+2+3+4 \\
15 & = & 1+2+3+4+5 \\
21 & = & 1+2+3+4+5+6 \\
28 & = & 1+2+3+4+5+6+7 \\
36 & = & 1+2+3+4+5+6+7+8
\end{array}
$$

+)

$$
1+3+6+10+15+21+28+36 \ = \begin{array}{l}
\ \ 1 \\
+1+2 \\
+1+2+3 \\
+1+2+3+4 \\
+1+2+3+4+5 \\
+1+2+3+4+5+6 \\
+1+2+3+4+5+6+7 \\
+1+2+3+4+5+6+7+8
\end{array}
$$

So we end up calculating

$$\boxed{\begin{array}{l} \quad 1 \\ +1+2 \\ +1+2+3 \\ +1+2+3+4 \\ +1+2+3+4+5 \\ +1+2+3+4+5+6 \\ +1+2+3+4+5+6+7 \\ +1+2+3+4+5+6+7+8 \end{array}} \quad \longleftarrow \quad \underline{\underline{\text{Let's call it}}} \ \underline{\underline{\boxed{\ x\ }}}$$

though it is still helpful to remember

$$\boxed{\ x\ } \ = \ \boxed{1+3+6+10+15+21+28+36}\ .$$

$\Big[\underline{\textbf{Solution}}\Big]$ $\Big($See ‘$\underline{\text{ Refresher \#1 }}$’ from “Supplement”.$\Big)$ Agree that x equals

$$\underset{\underline{\text{Box \#1}}}{\boxed{\begin{array}{lr} \ \ 1 & +2+3+4+5+6+7+8+9 \\ +1+2 & +3+4+5+6+7+8+9 \\ +1+2+3 & +4+5+6+7+8+9 \\ +1+2+3+4 & +5+6+7+8+9 \\ +1+2+3+4+5 & +6+7+8+9 \\ +1+2+3+4+5+6 & +7+8+9 \\ +1+2+3+4+5+6+7 & +8+9 \\ +1+2+3+4+5+6+7+8 & +9 \end{array}}} - \underset{\underline{\text{Box \#2}}}{\boxed{\begin{array}{r} 2+3+4+5+6+7+8+9 \\ +3+4+5+6+7+8+9 \\ +4+5+6+7+8+9 \\ +5+6+7+8+9 \\ +6+7+8+9 \\ +7+8+9 \\ +8+9 \\ +9 \end{array}}}$$

So we calculate Box #1 and Box #2 separately. $\big($Sounds familiar?$\big)$

First, Box #1 is calculated simply as

$$\left.\begin{array}{l}
\boxed{\ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\
\boxed{+\ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\
\boxed{+\ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\
\boxed{+\ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\
\boxed{+\ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\
\boxed{+\ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\
\boxed{+\ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} \\
\boxed{+\ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9}
\end{array}\right] \begin{array}{c}\text{eight}\\ \text{identical}\\ \text{boxes,}\\ \text{each}\\ \text{equals}\\ 45\end{array}$$

So

$$\underline{\text{ Box } \#1 \ } \ = \ 45 \cdot 8 \ = \ 360.$$

Meanwhile, Box #2 equals

$$\boxed{\ \ 2} \;
\boxed{\begin{array}{l}+\ 3\\+\ 3\end{array}} \;
\boxed{\begin{array}{l}+\ 4\\+\ 4\\+\ 4\end{array}} \;
\boxed{\begin{array}{l}+\ 5\\+\ 5\\+\ 5\\+\ 5\end{array}} \;
\boxed{\begin{array}{l}+\ 6\\+\ 6\\+\ 6\\+\ 6\\+\ 6\end{array}} \;
\boxed{\begin{array}{l}+\ 7\\+\ 7\\+\ 7\\+\ 7\\+\ 7\\+\ 7\end{array}} \;
\boxed{\begin{array}{l}+\ 8\\+\ 8\\+\ 8\\+\ 8\\+\ 8\\+\ 8\\+\ 8\end{array}} \;
\boxed{\begin{array}{l}+\ 9\\+\ 9\\+\ 9\\+\ 9\\+\ 9\\+\ 9\\+\ 9\\+\ 9\end{array}}$$

$$\begin{array}{l|ccccccccccccccc}
+) & & & & & & & & & & & & & & & \\ \hline
 & 2 & + & 6 & + & 12 & + & 20 & + & 30 & + & 42 & + & 56 & + & 72 \\
 & \| & & \| & & \| & & \| & & \| & & \| & & \| & & \| \\
 & 2\cdot\underline{\underline{1}} & & 2\cdot\underline{\underline{3}} & & 2\cdot\underline{\underline{6}} & & 2\cdot\underline{\underline{10}} & & 2\cdot\underline{\underline{15}} & & 2\cdot\underline{\underline{21}} & & 2\cdot\underline{\underline{28}} & & 2\cdot\underline{\underline{36}}
\end{array}$$

Ring a bell? In other words, Box #2 equals

$$2 \quad \Big(\underbrace{\,1 + 3 + 6 + 10 + 15 + 21 + 28 + 36\,}_{\substack{\| \\ x}}\Big)$$

(See ‘$\underline{\text{ Refresher \#2 }}$’ from “Supplement”.) Thus

$$\underline{\text{ Box \#2 }} \quad = \quad 2\,x.$$

Remember,

$$x \quad = \quad \underset{\substack{\| \\ 360}}{\underline{\text{ Box \#1 }}} \quad - \quad \underset{\substack{\| \\ 2\,x}}{\underline{\text{ Box \#2 }}}$$

So this reads

$$x \quad = \quad 360 \;-\; 2\,x.$$

This is readily solved as

$$x \;=\; 120.$$

(See ‘$\underline{\text{ Refresher \#3 }}$’ from “Supplement”.) This is the answer for part (8).

$\Big[$End of solution$\Big]$.

Wow. But you might say you can do it brute-force, and that is not really difficult. Yes, I agree. But this method actually has its own merits.

Namely, the list

(1) $1 \ =?$
(2) $1 + 3 \ =?$
(3) $1 + 3 + 6 \ =?$
(4) $1 + 3 + 6 + 10 \ =?$
(5) $1 + 3 + 6 + 10 + 15 \ =?$
(6) $1 + 3 + 6 + 10 + 15 + 21 \ =?$
(7) $1 + 3 + 6 + 10 + 15 + 21 + 28 \ =?$
(8) $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 \ =?$
(9) $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 \ =?$
(10) $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 \ =?$

continues endlessly. For example, it makes sense to ask "how much is the part (100)?" The part (100) actually looks exactly like this:

$$
\begin{array}{l}
(100)\ \ 1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 \\
\quad + 66 + 78 + 91 + 105 + 120 + 136 + 153 + 171 + 190 + 210 \\
\quad + 231 + 253 + 276 + 300 + 325 + 351 + 378 + 406 + 435 + 465 \\
\quad + 496 + 528 + 561 + 595 + 630 + 666 + 703 + 741 + 780 + 820 \\
\quad + 861 + 903 + 946 + 990 + 1035 + 1081 + 1128 + 1176 + 1225 + 1275 \\
\quad + 1326 + 1378 + 1431 + 1485 + 1540 + 1596 + 1653 + 1711 + 1770 + 1830 \\
\quad + 1891 + 1953 + 2016 + 2080 + 2145 + 2211 + 2278 + 2346 + 2415 + 2485 \\
\quad + 2556 + 2628 + 2701 + 2775 + 2850 + 2926 + 3003 + 3081 + 3160 + 3240 \\
\quad + 3321 + 3403 + 3486 + 3570 + 3655 + 3741 + 3828 + 3916 + 4005 + 4095 \\
\quad + 4186 + 4278 + 4371 + 4465 + 4560 + 4656 + 4753 + 4851 + 4950 + 5050 \quad =?
\end{array}
$$

Here, $\big($apparently$\big)$ the hundred numbers involved in part (100) are constructed one-by-one as

$$
\begin{array}{ccccc}
0 + 1 = 1, & 1 + 2 = 3, & 3 + 3 = 6, & 6 + 4 = 10, & 10 + 5 = 15, \\
15 + 6 = 21, & 21 + 7 = 28, & 28 + 8 = 36, & 36 + 9 = 45, & 45 + 10 = 55, \\
55 + 11 = 66, & 66 + 12 = 78, & 78 + 13 = 91, & 91 + 14 = 105, & 105 + 15 = 120, \\
120+16 = 136, & 136+17 = 153, & 153+18 = 171, & 171+19 = 190, & 190+20 = 210,
\end{array}
$$

and so on.

Do you want to do it brute-force? No, I bet you do not. Good news: The above method is applicable to do part (100). Indeed, part (100) equals

$$\boxed{\begin{array}{llll}
\quad 1 & & & \\
+1+2 & & & \\
+1+2+3 & & & \\
+1+2+3+4 & & & \\
+1+2+3+4+5 & & & \\
\quad\vdots & & \ddots & \\
+1+2+3+4+5+\cdots+99 & & & \\
+1+2+3+4+5+\cdots+99+100 & & &
\end{array}} \quad \longleftarrow \quad \underline{\underline{\text{Let's call it}}} \quad \underline{\underline{\boxed{x}}}$$

$$= \boxed{\begin{array}{lr}
\quad 1 & +2+3+4+5+6+\cdots+100+101 \\
+1+2 & +3+4+5+6+\cdots+100+101 \\
+1+2+3 & +4+5+6+\cdots+100+101 \\
+1+2+3+4 & +5+6+\cdots+100+101 \\
+1+2+3+4+5 & +6+\cdots+100+101 \\
\quad\vdots & \ddots \qquad \vdots \\
+1+2+3+4+5+\cdots+99 & +100+101 \\
+1+2+3+4+5+\cdots+99+100 & +101
\end{array}} \quad \underline{\text{Box \#1}}$$

$$- \boxed{\begin{array}{r}
2+3+4+5+6+\cdots+100+101 \\
+3+4+5+6+\cdots+100+101 \\
+4+5+6+\cdots+100+101 \\
+5+6+\cdots+100+101 \\
+6+\cdots+100+101 \\
\ddots \qquad \vdots \\
+100+101 \\
+101
\end{array}} \quad \underline{\text{Box \#2}}$$

First, Box #1 is calculated simply as

$$\left.\begin{array}{l}
\boxed{\ 1 + 2 + 3 + 4 + 5 + \cdots + 100 + 101} \\
\boxed{+\ 1 + 2 + 3 + 4 + 5 + \cdots + 100 + 101} \\
\boxed{+\ 1 + 2 + 3 + 4 + 5 + \cdots + 100 + 101} \\
\boxed{+\ 1 + 2 + 3 + 4 + 5 + \cdots + 100 + 101} \\
\qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
\boxed{+\ 1 + 2 + 3 + 4 + 5 + \cdots + 100 + 101}
\end{array}\right] \begin{array}{c}
\text{one} \\ \text{hundred} \\ \text{identical} \\ \text{boxes,} \\ \text{each} \\ \text{equals} \\ 5151
\end{array}$$

So

$$\underline{\text{Box } \#1} = 5151 \cdot 100 = 515100.$$

Meanwhile, Box #2:

$$\begin{array}{r}
2 + 3 + 4 + 5 + 6 + \cdots + 100 + 101 \\
+ 3 + 4 + 5 + 6 + \cdots + 100 + 101 \\
+ 4 + 5 + 6 + \cdots + 100 + 101 \\
+ 5 + 6 + \cdots + 100 + 101 \\
+ 6 + \cdots + 100 + 101 \\
\ddots \qquad\qquad \vdots \\
+ 100 + 101 \\
+ 101
\end{array}$$

simply equals $2\,x$. $\big($Explain why$\big)$. So

$$x = 515100 - 2\,x.$$

This is readily solved as

$$x = 171700.$$

So, to conclude:

$$
\begin{aligned}
(100)\quad & 1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 \\
& + 66 + 78 + 91 + 105 + 120 + 136 + 153 + 171 + 190 + 210 \\
& + 231 + 253 + 276 + 300 + 325 + 351 + 378 + 406 + 435 + 465 \\
& + 496 + 528 + 561 + 595 + 630 + 666 + 703 + 741 + 780 + 820 \\
& + 861 + 903 + 946 + 990 + 1035 + 1081 + 1128 + 1176 + 1225 + 1275 \\
& + 1326 + 1378 + 1431 + 1485 + 1540 + 1596 + 1653 + 1711 + 1770 + 1830 \\
& + 1891 + 1953 + 2016 + 2080 + 2145 + 2211 + 2278 + 2346 + 2415 + 2485 \\
& + 2556 + 2628 + 2701 + 2775 + 2850 + 2926 + 3003 + 3081 + 3160 + 3240 \\
& + 3321 + 3403 + 3486 + 3570 + 3655 + 3741 + 3828 + 3916 + 4005 + 4095 \\
& + 4186 + 4278 + 4371 + 4465 + 4560 + 4656 + 4753 + 4851 + 4950 + 5050
\end{aligned}
$$

$$
= \quad 171700 \ .
$$

Or, the same to say:

$$
\begin{array}{lcl}
\,1 & & \\
+\,1+2 & & \\
+\,1+2+3 & & \\
+\,1+2+3+4 & & \\
+\,1+2+3+4+5 & & \\
\ \vdots \qquad\qquad\qquad\qquad \ddots & & \\
+\,1+2+3+4+5+\cdots+99 & & \\
+\,1+2+3+4+5+\cdots+99+100 & \qquad = \qquad & 171700 \ .
\end{array}
$$

⋆ Now this method allows you to actually concoct the following formula:

Formula. Let n be a positive integer. Then

$$\boxed{1 + \underbrace{3}_{\substack{\| \\ 1+2}} + \underbrace{6}_{\substack{\| \\ 1+2+3}} + \underbrace{10}_{\substack{\| \\ 1+2+3+4}} + \cdots + \underbrace{\frac{1}{2}n\Big(n+1\Big)}_{\substack{\| \\ 1+2+3+4+\cdots+n}} = \frac{1}{6}\,n\Big(n+1\Big)\Big(n+2\Big).}$$

Formula paraphrased. Let n be a positive integer. Then

$$\boxed{\begin{aligned} & \;1 \\ & +1+2 \\ & +1+2+3 \\ & +1+2+3+4 \\ & +1+2+3+4+5 \\ & \;\vdots \qquad\qquad\qquad\qquad\qquad \ddots \\ & +1+2+3+4+5+\cdots+\Big(n-1\Big) \\ & +1+2+3+4+5+\cdots+\Big(n-1\Big)+n \\ & \qquad\qquad\qquad\qquad\qquad\qquad = \frac{1}{6}\,n\Big(n+1\Big)\Big(n+2\Big). \end{aligned}}$$

Here are some exercises.

Exercise 1.

(a)

$$1 + \underbrace{3}_{\substack{\| \\ 1+2}} + \underbrace{6}_{\substack{\| \\ 1+2+3}} + \underbrace{10}_{\substack{\| \\ 1+2+3+4}} + \cdots + \underbrace{210}_{\substack{\| \\ 1+2+3+4+\cdots+20}} \quad = \; ?$$

(b)

$$\begin{array}{ll}
\;\;1 & \\
+1+2 & \\
+1+2+3 & \\
+1+2+3+4 & \\
+1+2+3+4+5 & \\
\;\;\vdots \qquad\qquad\qquad\qquad \ddots & \\
+1+2+3+4+5+\cdots+999 & \\
+1+2+3+4+5+\cdots+999+1000 & \qquad = \; ?
\end{array}$$

[**Answers**]:

(a) By the first Formula (in the previous page), this is

$$\frac{1}{6} \cdot 20 \cdot 21 \cdot 22 \quad = \quad 1540.$$

(b) By the second Formula (in the previous page), this is

$$\frac{1}{6} \cdot 1000 \cdot 1001 \cdot 1002 \quad = \quad 167167000.$$