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§29. APPLICATIONS OF ANTIDERIVATIVES.

Today's theme is application of antiderivatives/indefinite integrals — not just random applications, but one specific application, in the context of the Bernoulli polynomials/numbers.

Two lectures ago (in "Review of Lectures – XXVII") we have practiced how to reconstruct the Bernoulli polynomials from the Bernoulli numbers. The pitfall of that method is it required us to refer to the table of Bernoulli numbers. So we needed to supply the method (algorithm) to generate the Bernoulli numbers separately. That part was covered in "Review of Lectures – XXVI", called 'Akiyama–Tanigawa diagram'. (See also "Extra Credit Homework – I".) Today, we learn a completely different method to reconstruct the Bernoulli polynomials/numbers, without relying on the 'Akiyama–Tanigawa diagram'. This is actually a perfect application of the concept of antiderivatives/indefinite integrals. So, our goal is to recover

$$B_1{}^\circ(x) = x - 1, \qquad\qquad B_1(x) = x - \frac{1}{2},$$

$$B_2{}^\circ(x) = x^2 - x, \qquad\qquad B_2(x) = x^2 - x + \frac{1}{6},$$

$$B_3{}^\circ(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \qquad\qquad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$

$$B_4{}^\circ(x) = x^4 - 2x^3 + x^2, \qquad\qquad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},$$

$$B_5{}^\circ(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \qquad B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x,$$

$$B_6{}^\circ(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2, \qquad B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42},$$

$$B_7{}^\circ(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x, \; B_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x,$$

$$\vdots$$

one by one, from the complete scratch.

1. Starting point of construction. $B_1^{\circ}(x)$.

We do not rely on any prior knowledge about the shape of Bernoulli polynomials/numbers, except

Initial conditions:

(i) $$B_k^{\circ}(0) = 0 \qquad \text{for all } k \text{ with} \qquad k \geq 2,$$

(ii) $$B_k^{\circ}(1) = 0 \qquad \text{for all } k \text{ with} \qquad k \geq 1,$$

and the following:

(1d) $$B_1^{\circ}(x) = x - 1.$$

$\left(\text{Note that there is no (1a), (1b) and (1c).}\right)$

We are going to follow the step-by-step procedure below. Those steps exhibit the same pattern, each step uses the outcome of the previous step. So after three or so steps, you get the firm idea how to proceed. As you can see, in (2d), (3d), and (4d), you see that $B_2^{\circ}(x)$, $B_3^{\circ}(x)$, $B_4^{\circ}(x)$ are recovered. At the same time, In (2c), (3c), and (4c), you see that n B_1, B_2, and B_3 are recovered.

2. Determining B_1 and $B_2^{\circ}(x)$.

Add B_1 to (1d), and thereby make it $B_1(x)$. At this point B_1 is an unknown constant.

$$B_1(x) = x - 1 + B_1.$$

Take its antiderivative:

$$\int B_1(x)\, dx = \frac{1}{2}x^2 - x + B_1 \cdot x + C.$$

This is $\dfrac{1}{2} B_2^{\circ}(x)$, so

$$(2a) \qquad \frac{1}{2} B_2{}^\circ (x) = \frac{1}{2} x^2 - x + B_1 \cdot x + C.$$

Substitute $x = 0$, and $x = 1$ into (2a) independently. Use $B_2{}^\circ (0) = 0$ and $B_2{}^\circ (1) = 0$ ('initial conditions'):

$$(2b) \qquad \begin{cases} 0 = \dfrac{1}{2} \cdot 0^2 - 0 + B_1 \cdot 0 + C, \\[2mm] 0 = \dfrac{1}{2} \cdot 1^2 - 1 + B_1 \cdot 1 + C. \end{cases}$$

The first of the two equations in (2b) reads $0 = C$. So $C = 0$. Taking this into account, the second of the two equations in (2b) becomes

$$0 = \frac{1}{2} - 1 + B_1.$$

Solve it:

$$B_1 = - \left(\frac{1}{2} - 1 \right)$$

$$= \frac{1}{2}.$$

So we just found B_1. Namely,

$$(2c) \qquad B_1 = \frac{1}{2}.$$

Substitute this and $C = 0$ back into (2a) above:

$$\frac{1}{2} B_2{}^\circ (x) = \frac{1}{2} x^2 - x + \frac{1}{2} x$$

$$= \frac{1}{2} x^2 - \frac{1}{2} x.$$

Multiply 2 to the both sides:

$$(2d) \qquad B_2{}^\circ (x) = x^2 - x.$$

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3. Determining B_2 and $B_3{}^\circ(x)$.

Add B_2 to (2d), and thereby make it $B_2(x)$. At this point B_2 is an unknown constant.

$$B_2(x) = x^2 - x + B_2.$$

Take its antiderivative:

$$\int B_2(x)\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + B_2 \cdot x + C.$$

This is $\frac{1}{3}B_3{}^\circ(x)$, so

(3a) $$\frac{1}{3}B_3{}^\circ(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + B_2 \cdot x + C.$$

Substitute $x = 0$, and $x = 1$ into (3a) independently. Use $B_3{}^\circ(0) = 0$ and $B_3{}^\circ(1) = 0$ ('initial conditions'):

(3b)
$$\begin{cases} 0 = \dfrac{1}{3} \cdot 0^3 - \dfrac{1}{2} \cdot 0^2 + B_2 \cdot 0 + C, \\[2mm] 0 = \dfrac{1}{3} \cdot 1^3 - \dfrac{1}{2} \cdot 1^2 + B_2 \cdot 1 + C. \end{cases}$$

The first of the two equations in (3b) reads $0 = C$. So $C = 0$. Taking this into account, the second of the two equations in (3b) becomes

$$0 = \frac{1}{3} - \frac{1}{2} + B_2.$$

Solve it:

$$B_2 = -\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{1}{6}.$$

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So we just found B_2. Namely,

(3c)
$$B_2 = \frac{1}{6}.$$

Substitute this and $C = 0$ back into (3a) above:

$$\frac{1}{3}B_3{}^\circ (x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x.$$

Multiply 3 to the both sides:

(3d)
$$B_3{}^\circ (x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

4. Determining B_3 and $B_4{}^\circ (x)$.

Add B_3 to (3d), and thereby make it $B_3 (x)$. At this point B_3 is an unknown constant.

$$B_3 (x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x + B_3.$$

Take its antiderivative:

$$\int B_3 (x)\, dx = \frac{1}{4}x^4 - \frac{1}{2}x^3 + \frac{1}{4}x^2 + B_3 \cdot x + C.$$

This is $\frac{1}{4}B_4{}^\circ (x),$ so

(4a)
$$\frac{1}{4}B_4{}^\circ (x) = \frac{1}{4}x^4 - \frac{1}{2}x^3 + \frac{1}{4}x^2 + B_3 \cdot x + C.$$

Substitute $x = 0,$ and $x = 1$ into (4a) independently.   Use $B_4{}^\circ (0) = 0$ and $B_4{}^\circ (1) = 0$ ('initial conditions'):

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(4b)
$$\begin{cases} 0 = \dfrac{1}{4} \cdot 0^4 - \dfrac{1}{2} \cdot 0^3 + \dfrac{1}{4} \cdot 0^2 + B_3 \cdot 0 + C, \\[4mm] 0 = \dfrac{1}{4} \cdot 1^4 - \dfrac{1}{2} \cdot 1^3 + \dfrac{1}{4} \cdot 1^2 + B_3 \cdot 1 + C. \end{cases}$$

The first of the two equations in (4b) reads $0 = C$. So $C = 0$. Taking this into account, the second of the two equations in (4b) becomes

$$0 = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} + B_3.$$

Solve it:

$$B_3 = -\left(\frac{1}{4} - \frac{1}{2} + \frac{1}{4} \right)$$
$$= 0.$$

So we just found B_3. Namely,

(4c)
$$B_3 = 0.$$

Substitute this and $C = 0$ back into (4a) above:

$$\frac{1}{4} B_4{}^\circ (x) = \frac{1}{4} x^4 - \frac{1}{2} x^3 + \frac{1}{4} x^2.$$

Multiply 4 to the both sides:

(4d)
$$B_4{}^\circ (x) = x^3 - 2x^3 + x^2.$$

⋆ I think you have seen enough so you can take it from here. Namely, add B_4 to (4d) and then integrate it. That is $\dfrac{1}{5}$ of $B_5{}^\circ (x)$. One of the initial conditions: $B_5{}^\circ (0) = 0$ tells you $C = 0$. The other initial condition: $B_5{}^\circ (1) = 0$ allows you to determine B_4. Substituting it in the expression of $B_5{}^\circ (x)$ previously obtained which involves B_4 yields the actual shape of $B_5{}^\circ (x)$. So, in principle, as you keep going this way, you recover as many Bernoulli polynomials/numbers as you want, though in practice the computation gets longer and longer as you go on. Now you can do the exercise below.

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Exercise 1.   Assume

(7d) $$B_7{}^\circ\left(x\right) \;=\; x^7 \;-\; \frac{7}{2}x^6 \;+\; \frac{7}{2}x^5 \;-\; \frac{7}{6}x^3 \;+\; \frac{1}{6}x,$$

and recover B_7 and $B_8{}^\circ(x)$.

$\boxed{\textbf{Solution}}$:   Add B_7 to (7d), and thereby make it $B_7\left(x\right)$. At this point B_7 is an unknown constant.

$$B_7\left(x\right) \;=\; x^7 \;-\; \frac{7}{2}x^6 \;+\; \frac{7}{2}x^5 \;-\; \frac{7}{6}x^3 \;+\; \frac{1}{6}x \;+\; B_7.$$

Take its antiderivative:

$$\int B_7\left(x\right) dx \;=\; \frac{1}{8}x^8 \;-\; \frac{1}{2}x^7 \;+\; \frac{7}{12}x^6 \;-\; \frac{7}{24}x^4 \;+\; \frac{1}{12}x^2 \;+\; B_7\cdot x \;+\; C.$$

This is $\dfrac{1}{8}B_8{}^\circ\left(x\right),$ so

(8a) $\dfrac{1}{8}B_8{}^\circ\left(x\right) \;=\; \dfrac{1}{8}x^8 \;-\; \dfrac{1}{2}x^7 \;+\; \dfrac{7}{12}x^6 \;-\; \dfrac{7}{24}x^4 \;+\; \dfrac{1}{12}x^2 \;+\; B_7\cdot x \;+\; C.$

Substitute $x = 0,$ and $x = 1$ into (8a) independently.   Use $B_8{}^\circ\left(0\right) = 0$ and $B_8{}^\circ\left(1\right) = 0$ ('initial conditions'):

(8b)
$$\begin{cases} 0 \;=\; \dfrac{1}{8}0^8 \;-\; \dfrac{1}{2}0^7 \;+\; \dfrac{7}{12}0^6 \;-\; \dfrac{7}{24}0^4 \;+\; \dfrac{1}{12}0^2 \;+\; B_7\cdot 0 \;+\; C. \\[3mm] 0 \;=\; \dfrac{1}{8}1^8 \;-\; \dfrac{1}{2}1^7 \;+\; \dfrac{7}{12}1^6 \;-\; \dfrac{7}{24}1^4 \;+\; \dfrac{1}{12}1^2 \;+\; B_7\cdot 1 \;+\; C. \end{cases}$$

The first of the two equations in (8b) reads $0 = C.$ So $C = 0.$ Taking this into account, the second of the two equations in (8b) becomes

$$0 \;=\; \frac{1}{8} \;-\; \frac{1}{2} \;+\; \frac{7}{12} \;-\; \frac{7}{24} \;+\; \frac{1}{12} \;+\; B_7.$$

Solve it:

$$B_7 = -\left(\frac{1}{8} - \frac{1}{2} + \frac{7}{12} - \frac{7}{24} + \frac{1}{12} \right)$$

$$= 0.$$

So we just found B_0. Namely,

(8c) $$B_7 = 0.$$

Substitute this and $C = 0$ back into (8a) above:

$$\frac{1}{8} B_8{}^\circ (x) = \frac{1}{8} x^8 - \frac{1}{2} x^7 + \frac{7}{12} x^6 - \frac{7}{24} x^4 + \frac{1}{12} x^2.$$

Multiply 8 to the both sides:

(8d) $$B_8{}^\circ (x) = x^8 - 4x^7 + \frac{14}{3} x^6 - \frac{7}{3} x^4 + \frac{2}{3} x^2.$$

Exercise 2. Assume

(12d) $$B_{12}{}^\circ (x) = x^{12} - 6 x^{11} + 11 x^{10} - \frac{33}{2} x^8 + 22 x^6 - \frac{33}{2} x^4 + 5 x^2,$$

and recover B_{12} and $B_{13}{}^\circ(x)$.

$\left[\underline{\textbf{Answer}} \right]$:

$$B_{12} = \frac{-691}{2730},$$

$$B_{13}(x) = x^{13} - \frac{13}{2} x^{12} + 13x^{11} - \frac{143}{6} x^9 + \frac{286}{7} x^7 - \frac{429}{10} x^5 + \frac{65}{3} x^3 - \frac{691}{210} x.$$