Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – XXVIII

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§28. Antiderivatives.

Today we learn two important concepts, which are closely related to each other. One is called

"antiderivative," or "anti-derivative."

As an operation, antiderivative is just "the inverse" of the derivative operation. So, conceptually this is not something that is entirely new. We all know that $2x$ is the derivative of x^2. Then we say x^2 is *an* antiderivative of $2x$. More generally:

Definition (Antiderivatives).

If a polynomial $\boxed{f(x)}$ is the derivative of another polynomial $\boxed{F(x)}$, then we call $\boxed{F(x)}$ *an* antiderivative of $\boxed{f(x)}$. *

So, basically, if

$$\boxed{F'(x) = f(x)},$$

then $\boxed{F(x)}$ is an antiderivative of $\boxed{f(x)}$.

Oh, is that it? Yes indeed, that is it. So this is easy, then. But you still need to see the following, to get a sense:

*Here I *did not* say "the" antiderivative. More on this below.

- **Most basic antiderivatives – I.**

Agree

$$\begin{aligned}
&\text{The derivative of } x \text{ is } 1.\\
&\text{The derivative of } x^2 \text{ is } 2x.\\
&\text{The derivative of } x^3 \text{ is } 3x^2.\\
&\text{The derivative of } x^4 \text{ is } 4x^3.\\
&\text{The derivative of } x^5 \text{ is } 5x^4.\\
&\text{The derivative of } x^6 \text{ is } 6x^5.\\
&\text{The derivative of } x^7 \text{ is } 7x^6.\\
&\text{The derivative of } x^8 \text{ is } 8x^7.\\
&\text{The derivative of } x^9 \text{ is } 9x^8.
\end{aligned}$$

These translate to

$$\begin{aligned}
&x \text{ is an antiderivative of } 1.\\
&x^2 \text{ is an antiderivative of } 2x.\\
&x^3 \text{ is an antiderivative of } 3x^2.\\
&x^4 \text{ is an antiderivative of } 4x^3.\\
&x^5 \text{ is an antiderivative of } 5x^4.\\
&x^6 \text{ is an antiderivative of } 6x^5.\\
&x^7 \text{ is an antiderivative of } 7x^6.\\
&x^8 \text{ is an antiderivative of } 8x^7.\\
&x^9 \text{ is an antiderivative of } 9x^8.
\end{aligned}$$

The above are useful, but what's on the next page turn out to be even more useful, because those basically tell you what antiderivatives of x^k are:

- **Most basic antiderivatives – II.**

$$\begin{aligned}
&\text{The derivative of } && x && \text{is } 1.\\
&\text{The derivative of } && \frac{1}{2}x^2 && \text{is } x.\\
&\text{The derivative of } && \frac{1}{3}x^3 && \text{is } x^2.\\
&\text{The derivative of } && \frac{1}{4}x^4 && \text{is } x^3.\\
&\text{The derivative of } && \frac{1}{5}x^5 && \text{is } x^4.\\
&\text{The derivative of } && \frac{1}{6}x^6 && \text{is } x^5.\\
&\text{The derivative of } && \frac{1}{7}x^7 && \text{is } x^6.\\
&\text{The derivative of } && \frac{1}{8}x^8 && \text{is } x^7.\\
&\text{The derivative of } && \frac{1}{9}x^9 && \text{is } x^8.
\end{aligned}$$

These translate to

$$\begin{aligned}
x &\quad \text{is an antiderivative of} \quad 1.\\
\frac{1}{2}x^2 &\quad \text{is an antiderivative of} \quad x.\\
\frac{1}{3}x^3 &\quad \text{is an antiderivative of} \quad x^2.\\
\frac{1}{4}x^4 &\quad \text{is an antiderivative of} \quad x^3.\\
\frac{1}{5}x^5 &\quad \text{is an antiderivative of} \quad x^4.\\
\frac{1}{6}x^6 &\quad \text{is an antiderivative of} \quad x^5.\\
\frac{1}{7}x^7 &\quad \text{is an antiderivative of} \quad x^6.\\
\frac{1}{8}x^8 &\quad \text{is an antiderivative of} \quad x^7.\\
\frac{1}{9}x^9 &\quad \text{is an antiderivative of} \quad x^8.
\end{aligned}$$

- **Antiderivative of a quantity that involves more than one term.**

Let's 'mix' what we have already learned.

Example 1. We know the fact

$$\text{The derivative of} \quad \frac{1}{3}x^3 + x \quad \text{is} \quad x^2 + 1.$$

Translation:

$$\frac{1}{3}x^3 + x \quad \text{is an antiderivative of} \quad x^2 + 1.$$

Example 2. We know the fact

$$\text{The derivative of} \quad \frac{2}{5}x^5 - 2x^3 + 4x^2 \quad \text{is} \quad 2x^4 - 6x^2 + 8x.$$

Translation:

$$\frac{2}{5}x^5 - 2x^3 + 4x^2 \quad \text{is an antiderivative of} \quad 2x^4 - 6x^2 + 8x.$$

- **Antiderivative of $f(x)$ is not unique. Two antiderivatives of $f(x)$ differ by a constant.**

Here is one important thing you should know.

"*If* $\boxed{F(x)}$ *is an antiderivative of* $\boxed{f(x)}$, *then*

$$F(x) + 1, \quad F(x) - 3, \quad F(x) + \frac{1}{2}, \quad F(x) - \sqrt{2}, \quad \textit{etc.}$$

too are all antiderivatives of the same $\boxed{f(x)}$."

- If you are not sure of what I mean by that, take a look at the following:

Example 3. We already know the fact that

$$\frac{1}{3}x^3 \text{ is an antiderivative of } x^2.$$

But then the following is also true:

$$\frac{1}{3}x^3 + 1 \text{ is an antiderivative of } x^2.$$

Moreover, the following are all true:

$$\frac{1}{3}x^3 + 4 \text{ is an antiderivative of } x^2.$$

$$\frac{1}{3}x^3 - 100 \text{ is an antiderivative of } x^2.$$

$$\frac{1}{3}x^3 + 256 \text{ is an antiderivative of } x^2.$$

$$\frac{1}{3}x^3 - \sqrt{2} \text{ is an antiderivative of } x^2.$$

$$\frac{1}{3}x^3 + e \text{ is an antiderivative of } x^2.$$

More generally, the following is true:

$$\frac{1}{3}x^3 + C \text{ is an antiderivative of } x^2,$$

as long as C denotes a constant (real number).

⋆ Now, the question you might ask is, is there any antiderivative of x^2, other than " $\frac{1}{3}x^3$ plus a constant"? The answer is <u>no</u> : $\frac{1}{3}x^3 + C$ $\Big($where C is a constant$\Big)$ <u>exhaust</u> all the antiderivatives of x^2. More generally:

- **Nature of antiderivatives.**

"*If* $\boxed{F(x)}$ *is one antiderivative of* $\boxed{f(x)}$, *then*

$\boxed{F(x) + C}$ $\Big($ C is an arbitrary constant real number $\Big)$

represents all the antiderivatives of $\boxed{f(x)}$".

⋆ Okay. So far what we have seen is

"(something) is an antiderivative of (something) ",

expressed in a full sentence. As for this, you may wonder if there is a convenient way to write it only using mathematical symbols. The answer is, 'yes, there is'. And that's in the second half of today's lecture, which starts now.

- **Indefinite Integrals.**

Like I said, today I plan to cover two items. We already took care of one. Now, the other one:

"indefinite integral."

The following definition summarizes it all:

Definition–Notation (Indefinite Integrals).

"Suspose $\boxed{F(x)}$ is an antiderivative of $\boxed{f(x)}$, then you write

$$\boxed{\int f(x)\,dx \quad = \quad F(x) \;+\; C}\,.$$

- We officially call the left-hand side in the above highlighted box the indefinite integral of $f(x)$, with respect to x. We call C the integral constant. Also, we call $f(x)$ inside $\int$ the integrand.

1. Always include "+ C" as a part of your answer.

If you look at the right-hand side of the above highlighted box, you see there is "+ C". This is necessary, due to the reason I already explained. Many students who saw the indefinite integrals for the first time tend to omit "+ C" as a part of the answer. So, please do not forget to include $+\,C$.

2. Always include the tail-end "dx" inside the integral symbol.

If you look at the left-hand side of the above highlighted box, you see there is this tail-end "dx". This tail-end dx is necessary. Students who saw the indefinite integrals for the first time tend to omit the tail-end dx. Please never do that. Every single time you omit the tail-end dx, I will correct you.

Example 4. $$\int x\,dx \quad = \quad \frac{1}{2}x^2 \;+\; C.$$

This means

"An antiderivative of x is $\frac{1}{2}x^2$."

Example 5. $$\int x^2\,dx \quad = \quad \frac{1}{3}x^3 \;+\; C.$$

This means

"An antiderivative of x^2 is $\frac{1}{3}x^3$."

- So, we no longer have to say verbally

"(something) is an antiderivative of (something) ",

even though we can, and often will, whenever we best see fit. What we have just done is we have created an alternative, shorter, way of expressing it, and it only uses mathematical symbols.

⋆ Let's familiarize ourselves with this new way of writing.

Exercise 1. Use the integral symbol to paraphrase

" An antiderivative of $4x^3$ is x^4."

[**Answer**]:

$$\int 4\,x^3\,dx \quad = \quad x^4 \; + \; C.$$

Exercise 2. Use the integral symbol to paraphrase

"An antiderivative of $8x^3 + 8$ is $2x^4 + 8x$."

[**Answer**]:

$$\int \Big(8\,x^3 \,+\, 8\Big)\,dx \quad = \quad 2\,x^4 \;+\; 8\,x \;+\; C.$$

- **Why is the adjective 'indefinite'? Is there a 'definite integral'?**

By the way, you may naturally wonder

" *Where is the adjective 'indefinite' come from? As in is there another kind of an integral, perhaps something to be called 'definite integral' ?*"

— Yes, you are absolutely right. There is indeed such a thing called 'definite integral'. We plan to cover it as well. 'Indefinite integrals' and 'definite integrals' are related, but today I am not going to talk about 'definite integrals'.

- **Find indefinite integrals.**

So far what we have covered: You are given a concrete $f(x)$. You are also given a concrete $F(x)$. You are told the fact that $F(x)$ is an antiderivative of $f(x)$. Then you have practiced how to write the same fact using the symbol

$$\int .$$

This was easy. But you must have been anticipating to see questions in the following format:

Format. You are given a concrete $f(x)$, but you are not given its antiderivative $F(x)$. Find

$$\int f(x)\ dx.$$

Questions in this exact format are what we are going to practice next. The most important rule that you have to remember is the following:

$$\begin{aligned}
\int 1\ dx &= x + C, \\
\int x\ dx &= \frac{1}{2}x^2 + C, \\
\int x^2\ dx &= \frac{1}{3}x^3 + C, \\
\int x^3\ dx &= \frac{1}{4}x^4 + C, \\
\int x^4\ dx &= \frac{1}{5}x^5 + C, \\
\int x^5\ dx &= \frac{1}{6}x^6 + C, \\
&\vdots
\end{aligned}$$

These are encapsulated in one line:

Rule (monomial integration rule).

$$\boxed{\int x^n \, dx \;=\; \frac{1}{n+1}\, x^{n+1} \;+\; C}$$

$\Big(n$ is an integer $\underline{\text{ constant }}$; $\quad n \geq 0\Big)$.

Example 6. Let's evaluate $\displaystyle\int \frac{1}{2} x^5 \, dx$.

Okay, here is how it goes. You see $\frac{1}{2}$ inside the integral symbol. Let's pretend that there is no $\frac{1}{2}$. Then the integral would be $\frac{1}{6}x^6$, well, $+C$, but let's worry about it later. So, $\frac{1}{6}x^6$. But that is when there is no $\frac{1}{2}$. In reality, there is $\frac{1}{2}$. So, the actual answer is $\frac{1}{2}$ times $\frac{1}{6}x^6$, that is, $\frac{1}{12}x^6$. And don't forget $+C$. In sum:

$$\int \frac{1}{2} x^5 \, dx \;=\; \frac{1}{12} x^6 + C.$$

Example 7. Let's evaluate $\displaystyle\int \frac{7}{4} x^8 \, dx$.

All right, the same deal. You see $\frac{7}{4}$ inside the integral symbol. Let's pretend that there is no $\frac{7}{4}$. Then the integral would be $\frac{1}{9}x^9$. But that is when there is no $\frac{7}{4}$. In reality, there is $\frac{7}{4}$. So, the actual answer is $\frac{7}{4}$ times $\frac{1}{9}x^9$, that is, $\frac{7}{36}x^9$. And don't forget $+C$. In sum:

$$\int \frac{7}{4} x^8 \, dx \;=\; \frac{7}{36} x^9 + C.$$

Example 8. Let's evaluate $\int \left(2x^4 + \frac{1}{4}x^2\right) dx.$

This is basically the same as the previous but notice that there are two terms in the integrand. The way it works is you do integrate term by term. First, $2x^4$ is integrated as $\frac{2}{5}x^5$. Next, $\frac{1}{4}x^2$ is integrated as $\frac{1}{12}x^3$. So, you construct the answer simply as

$$\frac{2}{5}x^5 + \frac{1}{12}x^3 + C.$$

In sum:

$$\int \left(2x^4 - \frac{1}{4}x^2\right) dx = \frac{2}{5}x^5 + \frac{1}{12}x^3 + C.$$

Example 9. Let's evaluate $\int \left(8x^7 - \frac{6}{5}x^5 + \frac{2}{7}x^3\right) dx.$

The same deal. Integrate term by term. Since

$$8x^7, \qquad \frac{6}{5}x^5 \qquad \text{and} \qquad \frac{2}{7}x^3$$

are integrated as

$$x^8, \qquad \frac{1}{5}x^6 \qquad \text{and} \qquad \frac{1}{14}x^4,$$

respectively, so

$$\int \left(8x^7 - \frac{6}{5}x^5 + \frac{2}{7}x^3\right) dx = x^8 - \frac{1}{5}x^6 + \frac{1}{14}x^4 + C.$$

Example 10. Let's evaluate $$\int \Big(x + 1\Big)\Big(x + 5\Big)\,dx.$$

You need to expand the integrand first. We know

$$\Big(x + 1\Big)\Big(x + 5\Big) = x^2 + 6x + 5.$$

Accordingly, the given integral becomes

$$\int \Big(x^2 + 6x + 5\Big)\,dx.$$

We know how to handle this. Namely,

$$\int \Big(x^2 + 6x + 5\Big)\,dx = \frac{1}{3}x^3 + 3x^2 + 5x + C.$$

Exercise 3. Evaluate

(1) $$\int 6\,x^3\,dx.$$

(2) $$\int \frac{1}{3}\,x^7\,dx.$$

(3) $$\int \Big(\frac{3}{4}\,x^2 + 3\Big)\,dx.$$

(4) $$\int \Big(\frac{8}{5}\,x^5 + 10\,x^4\Big)\,dx.$$

(5) $$\int \Big(x^2 + x + 1\Big)\,dx.$$

(6) $$\int \Big(x - 1\Big)\Big(x + 1\Big)\,dx.$$

(7) $$\int \Big(x + 1\Big)\Big(x + 3\Big)\,dx.$$

[Answers] :

(1) $$\int 6\,x^3\,dx = \frac{3}{2}\,x^4 + C.$$

(2) $$\int \frac{1}{3}\,x^7\,dx = \frac{1}{24}\,x^8 + C.$$

(3) $$\int \left(\frac{3}{4}\,x^2 + 3\right) dx = \frac{1}{4}\,x^3 + 3\,x + C.$$

(4) $$\int \left(\frac{8}{5}\,x^5 + 10\,x^4\right) dx = \frac{4}{15}\,x^6 + 2\,x^5 + C.$$

(5) $$\int \left(x^2 + x + 1\right) dx = \frac{1}{3}\,x^3 + \frac{1}{2}\,x^2 + x + C.$$

(6) $$\int \left(x - 1\right)\left(x + 1\right) dx = \frac{1}{3}\,x^3 - x + C.$$

(7) $$\int \left(x + 1\right)\left(x + 3\right) dx = \frac{1}{3}\,x^3 + 2\,x^2 + 3\,x + C.$$