Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XXVII

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§27. Bernoulli polynomials and numbers — II.

So far we know

Formula. Let n be a positive integer. Then

$$1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n^2 + \frac{1}{2}n\,,$$

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\,,$$

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2,$$

$$1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n\,,$$

$$1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2,$$

$$1^6 + 2^6 + 3^6 + 4^6 + \cdots + n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n\,.$$

You can extend this, namely, follow any of the three methods ('Strategy A', 'B' and 'C' in "Review of Lectures – XXVI"), and "in theory" you get the right formula for

$$1^k + 2^k + 3^k + 4^k + \cdots + n^k$$

for any positive integer k. (probably 'Strategy C' is computationally the easiest). In practice, the computation gets more and more involved as k becomes larger. The next page highlights this for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$:

$$1^k + 2^k + 3^k + 4^k + \cdots + n^k$$

equals

$$\frac{1}{2}n^2 + \frac{1}{2}n, \qquad (k=1)$$

$$\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n, \qquad (k=2)$$

$$\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2, \qquad (k=3)$$

$$\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n, \qquad (k=4)$$

$$\frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2, \qquad (k=5)$$

$$\frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n, \qquad (k=6)$$

$$\frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2, \qquad (k=7)$$

$$\frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n, \qquad (k=8)$$

$$\frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2, \qquad (k=9)$$

$$\frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n, \qquad (k=10)$$

$$\frac{1}{12}n^{12} + \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{5}{12}n^2, \qquad (k=11)$$

$$\frac{1}{13}n^{13} - \frac{1}{2}n^{12} + n^{11} - \frac{11}{6}n^9 + \frac{22}{7}n^7 - \frac{33}{10}n^5 + \frac{5}{3}n^3 - \frac{691}{2730}n\ . \qquad (k=12)$$

⋆ Here, like I said last time, the choice of the letter is kind of arbitrary. So, for no compelling reason, let's change the letter from n to x. At the same time, let's give each polynomial a name:

- $\boldsymbol{f_k(x)}$.

$$
\begin{aligned}
f_1(x) &= \frac{1}{2}x^2 + \frac{1}{2}x,\\
f_2(x) &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x,\\
f_3(x) &= \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2,\\
f_4(x) &= \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{1}{30}x,\\
f_5(x) &= \frac{1}{6}x^6 + \frac{1}{2}x^5 + \frac{5}{12}x^4 - \frac{1}{12}x^2,\\
f_6(x) &= \frac{1}{7}x^7 + \frac{1}{2}x^6 + \frac{1}{2}x^5 - \frac{1}{6}x^3 + \frac{1}{42}x,\\
f_7(x) &= \frac{1}{8}x^8 + \frac{1}{2}x^7 + \frac{7}{12}x^6 - \frac{7}{24}x^4 + \frac{1}{12}x^2,\\
f_8(x) &= \frac{1}{9}x^9 + \frac{1}{2}x^8 + \frac{2}{3}x^7 - \frac{7}{15}x^5 + \frac{2}{9}x^3 - \frac{1}{30}x,\\
f_9(x) &= \frac{1}{10}x^{10} + \frac{1}{2}x^9 + \frac{3}{4}x^8 - \frac{7}{10}x^6 + \frac{1}{2}x^4 - \frac{3}{20}x^2,\\
f_{10}(x) &= \frac{1}{11}x^{11} + \frac{1}{2}x^{10} + \frac{5}{6}x^9 - x^7 + x^5 - \frac{1}{2}x^3 + \frac{5}{66}x,\\
f_{11}(x) &= \frac{1}{12}x^{12} + \frac{1}{2}x^{11} + \frac{11}{12}x^{10} - \frac{11}{8}x^8 + \frac{11}{6}x^6 - \frac{11}{8}x^4 + \frac{5}{12}x^2,\\
f_{12}(x) &= \frac{1}{13}x^{13} - \frac{1}{2}x^{12} + x^{11} - \frac{11}{6}x^9 + \frac{22}{7}x^7 - \frac{33}{10}x^5 + \frac{5}{3}x^3 - \frac{691}{2730}x\ .
\end{aligned}
$$

⋆ Now, “not for nothing”, let’s differentiate these:

- **Derivatives of $f_k(x)$.**

$$
\begin{aligned}
{f_1}'(x) &= x + \frac{1}{2}, \\
{f_2}'(x) &= x^2 + x + \frac{1}{6}, \\
{f_3}'(x) &= x^3 + \frac{3}{2}\,x^2 + \frac{1}{2}\,x, \\
{f_4}'(x) &= x^4 + 2\,x^3 + x^2 - \frac{1}{30}, \\
{f_5}'(x) &= x^5 + \frac{5}{2}\,x^4 + \frac{5}{3}\,x^3 - \frac{1}{6}\,x, \\
{f_6}'(x) &= x^6 + 3\,x^5 + \frac{5}{2}\,x^4 - \frac{1}{2}\,x^2 + \frac{1}{42}, \\
{f_7}'(x) &= x^7 + \frac{7}{2}\,x^6 + \frac{7}{2}\,x^5 - \frac{7}{6}\,x^3 + \frac{1}{6}\,x, \\
{f_8}'(x) &= x^8 + 4\,x^7 + \frac{14}{3}\,x^6 - \frac{7}{3}\,x^4 + \frac{2}{3}\,x^2 - \frac{1}{30}, \\
{f_9}'(x) &= x^9 + \frac{9}{2}\,x^8 + 6\,x^7 - \frac{21}{5}\,x^5 + 2\,x^3 - \frac{3}{10}\,x, \\
{f_{10}}'(x) &= x^{10} + 5\,x^9 + \frac{15}{2}\,x^8 - 7\,x^6 + 5\,x^4 - \frac{3}{2}\,x^2 + \frac{5}{66}, \\
{f_{11}}'(x) &= x^{11} + \frac{11}{2}\,x^{10} + \frac{55}{6}\,x^9 - 11\,x^7 + 11\,x^5 - \frac{11}{2}\,x^3 + \frac{5}{6}\,x, \\
{f_{12}}'(x) &= x^{12} + 6\,x^{11} + 11\,x^{10} - \frac{33}{2}\,x^8 + 22\,x^6 - \frac{33}{2}\,x^4 + 5\,x^2 - \frac{691}{2730}.
\end{aligned}
$$

What do you notice? Is there anything that stands out?

Maybe $f_{11}'(x)$. Indeed, $f_{11}'(x)$ has got a lot of 11s (and there is also 55 which is divisible by 11). Let's get rid of those 11s, by way of dividing the entire polynomial by 11:

$$\frac{1}{11} f_{11}'(x) = \frac{1}{11} x^{11} + \frac{1}{2} x^{10} + \frac{5}{6} x^9 - x^7 + x^5 - \frac{1}{2} x^3 + \frac{5}{66} x.$$

Now, this may be abrupt but let's go back to two pages ago, the list of $f_k(x)$. The same polynomial

$$\frac{1}{11} x^{11} + \frac{1}{2} x^{10} + \frac{5}{6} x^9 - x^7 + x^5 - \frac{1}{2} x^3 + \frac{5}{66} x$$

is actually found in that list. Namely, it is $f_{10}(x)$. So,

$$\frac{1}{11} f_{11}'(x) = f_{10}(x).$$

Now, is that a coincidence? Namely, do you expect that $\frac{1}{k} f_k'(x)$ and $f_{k-1}(x)$ always coincide? Well, let's compare $\frac{1}{10} f_{10}'(x)$ with $f_9(x)$:

$$\frac{1}{10} f_{10}'(x) = \frac{1}{10} x^{10} + \frac{1}{2} x^9 + \frac{3}{4} x^8 - \frac{7}{10} x^6 + \frac{1}{2} x^4 - \frac{3}{20} x^2 + \frac{1}{132}.$$

$$f_9(x) = \frac{1}{10} x^{10} + \frac{1}{2} x^9 + \frac{3}{4} x^8 - \frac{7}{10} x^6 + \frac{1}{2} x^4 - \frac{3}{20} x^2.$$

These are very close, though are not identical. Namely, the only difference is the last constant term. This constant $\frac{1}{132}$ is actually written as $\frac{1}{10} B_{10}$. So

$$B_{10} = \frac{5}{66}.$$

B_{10} is called <u>the tenth Bernoulli number</u>. Now, previously $\frac{1}{11} f_{11}'(x)$ and $f_{10}(x)$ coincided. That simply means that the eleventh Bernoulli number is just zero: $B_{11} = 0$. This way we can extract Bernoulli numbers $B_1, B_2, B_3, B_4, \cdots$ from $f_1(x), f_2(x), f_3(x), f_4(x), \cdots$. I attached the table of the first forty six (46) Bernoulli numbers (see page 9). Now, the polynomials in the previous page (with the shift of $x \mapsto x-1$) are called Bernoulli polynomials:

- **(a) Bernoulli polynomials $B_k(x)$.**

$$B_1(x) = f_1{}'\Big(x-1\Big) = x - \frac{1}{2},$$

$$B_2(x) = f_2{}'\Big(x-1\Big) = x^2 - x + \frac{1}{6},$$

$$B_3(x) = f_3{}'\Big(x-1\Big) = x^3 - \frac{3}{2}\,x^2 + \frac{1}{2}\,x,$$

$$B_4(x) = f_4{}'\Big(x-1\Big) = x^4 - 2\,x^3 + x^2 - \frac{1}{30},$$

$$B_5(x) = f_5{}'\Big(x-1\Big) = x^5 - \frac{5}{2}\,x^4 + \frac{5}{3}\,x^3 - \frac{1}{6}\,x,$$

$$B_6(x) = f_6{}'\Big(x-1\Big) = x^6 - 3\,x^5 + \frac{5}{2}\,x^4 - \frac{1}{2}\,x^2 + \frac{1}{42},$$

$$B_7(x) = f_7{}'\Big(x-1\Big) = x^7 - \frac{7}{2}\,x^6 + \frac{7}{2}\,x^5 - \frac{7}{6}\,x^3 + \frac{1}{6}\,x,$$

$$B_8(x) = f_8{}'\Big(x-1\Big) = x^8 - 4\,x^7 + \frac{14}{3}\,x^6 - \frac{7}{3}\,x^4 + \frac{2}{3}\,x^2 - \frac{1}{30},$$

$$B_9(x) = f_9{}'\Big(x-1\Big) = x^9 - \frac{9}{2}\,x^8 + 6\,x^7 - \frac{21}{5}\,x^5 + 2\,x^3 - \frac{3}{10}\,x,$$

$$B_{10}(x) = f_{10}{}'\Big(x-1\Big) = x^{10} - 5\,x^9 + \frac{15}{2}\,x^8 - 7\,x^6 + 5\,x^4 - \frac{3}{2}\,x^2 + \frac{5}{66},$$

$$B_{11}(x) = f_{11}{}'\Big(x-1\Big) = x^{11} - \frac{11}{2}\,x^{10} + \frac{55}{6}\,x^9 - 11\,x^7 + 11\,x^5 - \frac{11}{2}\,x^3 - \frac{5}{6}\,x,$$

$$B_{12}(x) = f_{12}{}'\Big(x-1\Big) = x^{12} - 6\,x^{11} + 11\,x^{10} - \frac{33}{2}\,x^8 + 22\,x^6 - \frac{33}{2}\,x^4 + 5\,x^2 - \frac{691}{2730}.$$

⋆ Compare these with $f_k{}'(x)$ two pages ago. The only difference is the sign for the second term. (Think about the reason why.) The constant terms of these polynomials are the Bernoulli numbers, with the only exception of $B_1(x)$. The constant term of $B_1(x)$ is $-\dfrac{1}{2}$ whereas $B_1 = \dfrac{1}{2}$. Meanwhile, we also define

- **(b) Truncated Bernoulli polynomials $B_k{}^\circ(x)$.**

$$B_1{}^\circ(x) = 1 f_0\Big(x-1\Big) = x-1,$$

$$B_2{}^\circ(x) = 2 f_1\Big(x-1\Big) = x^2-x,$$

$$B_3{}^\circ(x) = 3 f_2\Big(x-1\Big) = x^3-\frac{3}{2}x^2+\frac{1}{2}x,$$

$$B_4{}^\circ(x) = 4 f_3\Big(x-1\Big) = x^4-2x^3+x^2,$$

$$B_5{}^\circ(x) = 5 f_4\Big(x-1\Big) = x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x,$$

$$B_6{}^\circ(x) = 6 f_5\Big(x-1\Big) = x^6-3x^5+\frac{5}{2}x^4-\frac{1}{2}x^2,$$

$$B_7{}^\circ(x) = 7 f_6\Big(x-1\Big) = x^7-\frac{7}{2}x^6+\frac{7}{2}x^5-\frac{7}{6}x^3+\frac{1}{6}x,$$

$$B_8{}^\circ(x) = 8 f_7\Big(x-1\Big) = x^8-4x^7+\frac{14}{3}x^6-\frac{7}{3}x^4+\frac{2}{3}x^2,$$

$$B_9{}^\circ(x) = 9 f_8\Big(x-1\Big) = x^9-\frac{9}{2}x^8+6x^7-\frac{21}{5}x^5+2x^3-\frac{3}{10}x,$$

$$B_{10}{}^\circ(x) = 10 f_9\Big(x-1\Big) = x^{10}-5x^9+\frac{15}{2}x^8-7x^6+5x^4-\frac{3}{2}x^2,$$

$$B_{11}{}^\circ(x) = 11 f_{10}\Big(x-1\Big) = x^{11}-\frac{11}{2}x^{10}+\frac{55}{6}x^9-11x^7+11x^5-\frac{11}{2}x^3-\frac{5}{6}x,$$

$$B_{12}{}^\circ(x) = 12 f_{11}\Big(x-1\Big) = x^{12}-6x^{11}+11x^{10}-\frac{33}{2}x^8+22x^6-\frac{33}{2}x^4+5x^2.$$

The circle '$\circ$' in the notation $B_k{}^\circ(x)$ indicates that it is different from $B_k(x)$.

At first it might be slightly confusing to have two different versions of Bernoulli polynomials, one is original (non-truncated), and one is truncated. Now, the truth is, it is actually convenient to have both. The difference between $B_k(x)$ and $B_k^{\circ}(x)$ is only the constant term, and that constant is nothing but the k-th Bernoulli number B_k. So

$$\boxed{B_k(x) - B_k^{\circ}(x) = B_k.}$$

Or the same to say

$$\boxed{B_k(x) = B_k^{\circ}(x) + B_k.}$$

- **Initial conditions.**

The truncated Bernoulli polynomials satisfy some basic properties called initial conditions:

Initial conditions:

(i) $$B_k^{\circ}(0) = 0 \qquad \text{for all } k \text{ with} \qquad k \geq 2,$$

(ii) $$B_k^{\circ}(1) = 0 \qquad \text{for all } k \text{ with} \qquad k \geq 1.$$

The condition (i) simply means that $B_2^{\circ}(x),\ B_3^{\circ}(x),\ B_4^{\circ}(x),\ B_5^{\circ}(x),\ \cdots$ has no constant terms. These (i) and (ii) are important because they actually enable us to reconstruct the Bernoulli numbers and Bernoulli polynomials. To explain it, we need a new concept, called antiderivatives, which we will cover in the next lecture.

- Finally, the Bernoulli polynomials possess the following symmetry:

Symmetry:

$$B_k(1-x) = -B_k(x) \qquad \left(\text{when } k \text{ is odd}\right),$$

$$B_k(1-x) = B_k(x) \qquad \left(\text{when } k \text{ is even}\right).$$

- **Table of the first forty six (46) Bernoulli numbers.**

$$
\begin{array}{lll}
B_1 = \frac{1}{2}, & & B_2 = \frac{1}{6}, \\
B_3 = 0, & & B_4 = \frac{-1}{30}, \\
B_5 = 0, & & B_6 = \frac{1}{42}, \\
B_7 = 0, & & B_8 = \frac{-1}{30}, \\
B_9 = 0, & & B_{10} = \frac{5}{66}, \\
B_{11} = 0, & & B_{12} = \frac{-691}{2730}, \\
B_{13} = 0, & & B_{14} = \frac{7}{6}, \\
B_{15} = 0, & & B_{16} = \frac{-3617}{510}, \\
B_{17} = 0, & & B_{18} = \frac{43867}{798}, \\
B_{19} = 0, & & B_{20} = \frac{-174611}{330}, \\
B_{21} = 0, & & B_{22} = \frac{854513}{138}, \\
B_{23} = 0, & & B_{24} = \frac{-236364091}{2730}, \\
B_{25} = 0, & & B_{26} = \frac{8553103}{6}, \\
B_{27} = 0, & & B_{28} = \frac{-23749461029}{870}, \\
B_{29} = 0, & & B_{30} = \frac{8615841276005}{14322}, \\
B_{31} = 0, & & B_{32} = \frac{-7709321041217}{510}, \\
B_{33} = 0, & & B_{34} = \frac{2577687858367}{6}, \\
B_{35} = 0, & & B_{36} = \frac{-26315271553053477373}{1919190}, \\
B_{37} = 0, & & B_{38} = \frac{2929993913841559}{6}, \\
B_{39} = 0, & & B_{40} = \frac{-261082718496449122051}{13530}, \\
B_{41} = 0, & & B_{42} = \frac{1520097643918070802691}{1806}, \\
B_{43} = 0, & & B_{44} = \frac{-27833269579301024235023}{690}, \\
B_{45} = 0, & & B_{46} = \frac{596451111593912163277961}{282}.
\end{array}
$$

- **How do we reconstruct Bernoulli polynomials from Bernoulli numbers?**

If you have followed "Review of Lectures – XXVI", 'Strategy C', and also what we have covered today, then you realize that the k-th Bernoulli polynomial $B_k(x)$ is reconstructed from

$$B_1,\ B_2,\ B_3,\ B_4,\ \cdots\ ,\ B_k,$$

as follows:

Process. First, binomially expand

$$\left(x - B\right)^k.$$

Then 'lower' the exponents for B, namely, replace B^1 with B_1; B^2 with B_2; B^3 with B_3, and so on. The outcome is $B_k\left(x\right)$.

If you want to get $B_k{}^\circ\left(x\right)$, then just subtract B_k from $B_k\left(x\right)$.

⋆ The reason you see the negative sign inside the parenthesis $\left(x - B\right)^k$ instead of the positive sign is essentially due to the shift $x \mapsto x-1$, made in the definition of $B_k\left(x\right)$ in page 6.

Example 1. Let's reconstrct $B_6\left(x\right)$ and $B_6{}^\circ\left(x\right)$ using this method, and using the table in page 9.

Step 1. Binomially expand

$$\left(x - B\right)^6.$$

The result is

$$x^6 - \binom{6}{1}B^1x^5 + \binom{6}{1}B^2x^4 - \binom{6}{1}B^3x^3 + \binom{6}{1}B^4x^2 - \binom{6}{1}B^5x + B^6.$$

Step 2. Lower the exponents for B:

$$x^6 - \binom{6}{1} B_1 x^5 + \binom{6}{2} B_2 x^4 - \binom{6}{3} B_3 x^3 + \binom{6}{4} B_4 x^2 - \binom{6}{5} B_5 x + B_6.$$

Step 3. Throw concrete numbers for B_1, B_2, B_3, B_1, B_5 and B_6, using the table in page 9, and also throw concrete numbers for the binomial coefficients:

$$x^6 - 6 \cdot \frac{1}{2} x^5 + 15 \cdot \frac{1}{6} x^4 - 20 \cdot 0 x^3 + 15 \cdot \frac{-1}{30} x^2 - 6 \cdot 0 x + \cdot \frac{1}{42}$$

$$= x^6 - 3x^5 + \frac{5}{2} x^4 - \frac{1}{2} x^2 + \frac{1}{42}.$$

This is $B_6(x)$. In sum:

$$B_6(x) = x^6 - 3x^5 + \frac{5}{2} x^4 - \frac{1}{2} x^2 + \frac{1}{42}.$$

Step 4. As for $B_6{}^\circ(x)$, just frop the constant term, so

$$B_6{}^\circ(x) = x^6 - 3x^5 + x^4 - \frac{1}{2} x^2.$$

⋆ You can write your answer in the following way:

$$\begin{aligned} B_6(x) &= x^6 - \binom{6}{1} B_1 x^5 + \binom{6}{2} B_2 x^4 - \binom{6}{3} B_3 x^3 + \binom{6}{4} B_4 x^2 - \binom{6}{5} B_5 x + B_6 \\ &= x^6 - 6 \cdot \frac{1}{2} x^5 + 15 \cdot \frac{1}{6} x^4 - 20 \cdot 0 x^3 + 15 \cdot \frac{-1}{30} x^2 - 6 \cdot 0 x + \frac{1}{42} \\ &= x^6 - 3x^5 + x^4 - \frac{1}{2} x^2 + \frac{1}{42}, \end{aligned}$$

$$B_6{}^\circ(x) = x^6 - 3x^5 + x^4 - \frac{1}{2} x^2.$$

Exercise 1. Mimic Example 1 above and reconstruct each of

(1) $B_9(x)$, $B_9{}^\circ(x)$. (2) $B_{12}(x)$, $B_{12}{}^\circ(x)$.

$\left[\underline{\textbf{Solutions}}\right]$**:**

$$\text{(1)}\quad B_9(x) = x^9 - \binom{9}{1}B_1x^8 + \binom{9}{2}B_2x^7 - \binom{9}{3}B_3x^6 + \binom{9}{4}B_4x^5 - \binom{9}{5}B_5x^4$$

$$+\binom{9}{6}B_6x^3 - \binom{9}{7}B_7x^2 + \binom{9}{8}B_8x - B_9$$

$$= x^9 - 9\cdot\frac{1}{2}x^8 + 36\cdot\frac{1}{6}x^7 - 84\cdot 0x^6 + 126\cdot\frac{-1}{30}x^5$$

$$-126\cdot 0x^4 + 84\cdot\frac{1}{42}x^3 - 36\cdot 0x^2 + 9\cdot\frac{-1}{30}x - 0$$

$$= x^9 - \frac{9}{2}x^8 + 6x^7 - \frac{21}{5}x^5 + 2x^3 - \frac{3}{10}x.$$

$$B_9{}^\circ(x) = x^9 - \frac{9}{2}x^8 + 6x^7 - \frac{21}{5}x^5 + 2x^3 - \frac{3}{10}x.$$

$$\text{(2)}\quad B_{12}(x) = x^{12} - \binom{12}{1}B_1x^{12} + \binom{12}{2}B_2x^{11} - \binom{12}{3}B_3x^{10} + \binom{12}{4}B_4x^9$$

$$-\binom{12}{5}B_5x^8 + \binom{12}{6}Bx^7 - \binom{12}{7}B_7x^6 + \binom{12}{8}B_8x^5$$

$$-\binom{12}{7}B_9x^4 + \binom{12}{8}B_{10}x^3 - \binom{12}{8}B_{11}x^2 + B_{12}$$

$$= x^{12} - 12\cdot\frac{1}{2}x^{11} + 66\cdot\frac{1}{6}x^{10} - 220\cdot 0x^9 + 495\cdot\frac{-1}{30}x^8$$

$$-792\cdot 0x^7 + 924\cdot\frac{1}{42}x^6 - 792\cdot 0x^5 + 495\cdot\frac{-1}{30}x^4 - 220\cdot 0x^3$$

$$+66\cdot\frac{5}{66}x^2 - 12\cdot 0x + \frac{-691}{2730}$$

$$= x^{12} - 6\,x^{11} + 11\,x^{10} - \frac{33}{2}\,x^{8} + 22\,x^{6} - \frac{33}{2}\,x^{4} + 5\,x^{2} - \frac{691}{2730}.$$

$$B_{12}{}^{\circ}\left(x\right) = x^{12} - 6\,x^{11} + 11\,x^{10} - \frac{33}{2}\,x^{8} + 22\,x^{6} - \frac{33}{2}\,x^{4} + 5\,x^{2}.$$