Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XXVI

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§26. Bernoulli polynomials and numbers — I.

Back in "Review of Lectures – II' (way back), we have studied

(1) $1 \;=?$

(2) $1 + 2 \;=?$

(3) $1 + 2 + 3 \;=?$

(4) $1 + 2 + 3 + 4 \;=?$

(5) $1 + 2 + 3 + 4 + 5 \;=?$

(6) $1 + 2 + 3 + 4 + 5 + 6 \;=?$

(7) $1 + 2 + 3 + 4 + 5 + 6 + 7 \;=?$

(8) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \;=?$

(9) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \;=?$

$\vdots \qquad\qquad \ddots$

Then back in "Review of Lectures – XX", and "XXI", we have studied

(1) $1^2 \;=?$

(2) $1^2 + 2^2 \;=?$

(3) $1^2 + 2^2 + 3^2 \;=?$

(4) $1^2 + 2^2 + 3^2 + 4^2 \;=?$

(5) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 \;=?$

(6) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 \;=?$

(7) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \;=?$

(8) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 \;=?$

(9) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 \;=?$

$\vdots \qquad\qquad \ddots$

and

(1) $1^3 \; =?$

(2) $1^3 + 2^3 \; =?$

(3) $1^3 + 2^3 + 3^3 \; =?$

(4) $1^3 + 2^3 + 3^3 + 4^3 \; =?$

(5) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 \; =?$

(6) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 \; =?$

(7) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 \; =?$

(8) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 \; =?$

(9) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 \; =?$

$$\vdots \qquad\qquad\qquad\qquad \ddots$$

Let's recall the formulas for those:

Formula. Let n be a positive integer. Then

$$\boxed{1 + 2 + 3 + 4 + \cdots + n \quad = \quad \frac{1}{2}\, n\Big(n+1\Big)}\,,$$

$$\boxed{1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 \quad = \quad \frac{1}{6}\, n\Big(n+1\Big)\Big(2n+1\Big)}\,,$$

$$\boxed{1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 \quad = \quad \frac{1}{4}\, n^2\Big(n+1\Big)^2}\,.$$

Agree that the right-hand side of each of these formulas is a polynomial:

$$\frac{1}{2}\, n\Big(n+1\Big), \qquad \frac{1}{6}\, n\Big(n+1\Big)\Big(2n+1\Big), \qquad \frac{1}{4}\, n^2\Big(n+1\Big)^2.$$

These are polynomials <u>in n</u> to be precise.

⋆ In the last few lectures, we have always used the letter 'x' instead of 'n' for the variable of a polynomial. Let's not worry about that minor difference. There is no strict rule as to the choice of the letter for the variable of a polynomial. Typically, x, t, z, u, s, *etc.* are popular choices for the variable of a polynomial. Letters like o and i should be avoided $\big($$o$ is confused with 0, whereas i often denotes the unit imaginary number $\sqrt{-1}$, which we have briefly touched in "Review fo Lectures – XV"$\big)$.

Now, I am interested in expanding

$$\frac{1}{2}\, n\Big(n+1\Big), \qquad \frac{1}{6}\, n\Big(n+1\Big)\Big(2n+1\Big) \quad \text{and} \quad \frac{1}{4}\, n^2\Big(n+1\Big)^2$$

each. Let's just perform:

$$\begin{aligned} \frac{1}{2}\, n\Big(n+1\Big) &= \frac{1}{2}\Big(n^2 + n\Big) \\ &= \frac{1}{2}n^2 + \frac{1}{2}n, \end{aligned}$$

$$\begin{aligned} \frac{1}{6}\, n\Big(n+1\Big)\Big(2n+1\Big) &= \frac{1}{6}\Big(n^2+n\Big)\Big(2n+1\Big) \\ &= \frac{1}{6}\Big(2n^3 + n^2 + 2n^2 + n\Big) \\ &= \frac{1}{6}\Big(2n^3 + 3n^2 + n\Big) \\ &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n, \end{aligned}$$

$$\begin{aligned} \frac{1}{4}\, n^2\Big(n+1\Big)^2 &= \frac{1}{4}\, n^2\Big(n^2 + 2n + 1\Big) \\ &= \frac{1}{4}\Big(n^4 + 2n^3 + n^2\Big) \\ &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2. \end{aligned}$$

Accordingly, we can paraphrase the formulas above:

Formula paraphrased. Let n be a positive integer. Then

$$\boxed{1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n^2 + \frac{1}{2}n},$$

$$\boxed{1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n},$$

$$\boxed{1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2}.$$

Don't you want to create formulas for

$$\begin{gathered} 1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4, \\ 1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5, \\ 1^6 + 2^6 + 3^6 + 4^6 + \cdots + n^6, \\ \vdots \end{gathered}$$

There are basically two mutually alternative strategies.

Strategy A. Find the numbers in the boxes:

$$\begin{aligned}
&\boxed{6}\binom{n+2}{3} - \boxed{6}\binom{n+1}{2} + \boxed{1}\binom{n}{1} = n^3, \\
&\boxed{}\binom{n+3}{4} - \boxed{}\binom{n+2}{3} + \boxed{}\binom{n+1}{2} - \boxed{}\binom{n}{1} = n^4, \\
&\boxed{}\binom{n+4}{5} - \boxed{}\binom{n+3}{4} + \boxed{}\binom{n+2}{3} - \boxed{}\binom{n+1}{2} + \boxed{}\binom{n}{1} = n^5, \\
&\boxed{}\binom{n+5}{6} - \boxed{}\binom{n+4}{5} + \boxed{}\binom{n+3}{4} - \boxed{}\binom{n+2}{3} + \boxed{}\binom{n+1}{2} - \boxed{}\binom{n}{1} = n^6, \\
&\qquad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddots
\end{aligned}$$

As for this, the following tweak of Pascal gives us the answer:

$$
\begin{array}{cl}
\boxed{1} & \text{(row 0)} \\
{}_2/ \quad \backslash_1 & \\
\boxed{2} \quad \boxed{1} & \text{(row 1)} \\
{}_3/ \quad {}_2\backslash \quad /_2 \quad \backslash_1 & \\
\boxed{6} \quad \boxed{6} \quad \boxed{1} & \text{(row 2)} \\
{}_4/ \quad {}_3\backslash \quad /_3 \quad {}_2\backslash \quad /_2 \quad \backslash_1 & \\
\boxed{24} \quad \boxed{36} \quad \boxed{14} \quad \boxed{1} & \text{(row 3)} \\
{}_5/ \quad {}_4\backslash \quad /_4 \quad {}_3\backslash \quad /_3 \quad {}_2\backslash \quad /_2 \quad \backslash_1 & \\
\boxed{120} \quad \boxed{240} \quad \boxed{150} \quad \boxed{30} \quad \boxed{1} & \text{(row 4)} \\
{}_6/ \quad {}_5\backslash \quad /_5 \quad {}_4\backslash \quad /_4 \quad {}_3\backslash \quad /_3 \quad {}_2\backslash \quad /_2 \quad \backslash_1 & \\
\boxed{720} \quad \boxed{1800} \quad \boxed{1560} \quad \boxed{540} \quad \boxed{62} \quad \boxed{1} & \text{(row 5)} \\
{}_7/ \quad {}_6\backslash \quad /_6 \quad {}_5\backslash \quad /_5 \quad {}_4\backslash \quad /_4 \quad {}_3\backslash \quad /_3 \quad {}_2\backslash \quad /_2 \quad \backslash_1 & \\
\boxed{5040} \quad \boxed{15120} \quad \boxed{16800} \quad \boxed{8400} \quad \boxed{1806} \quad \boxed{126} \quad \boxed{1} & \text{(row 6)} \\
{}_8/ \quad {}_7\backslash \quad /_7 \quad {}_6\backslash \quad /_6 \quad {}_5\backslash \quad /_5 \quad {}_4\backslash \quad /_4 \quad {}_3\backslash \quad /_3 \quad {}_2\backslash \quad /_2 \quad \backslash_1 &
\end{array}
$$

Namely,

$$\boxed{6}\binom{n+2}{3} - \boxed{6}\binom{n+1}{2} + \boxed{1}\binom{n}{1} = n^3,$$

$$\boxed{24}\binom{n+3}{4} - \boxed{36}\binom{n+2}{3} + \boxed{14}\binom{n+1}{2} - \boxed{1}\binom{n}{1} = n^4,$$

$$\boxed{120}\binom{n+4}{5} - \boxed{240}\binom{n+3}{4} + \boxed{150}\binom{n+2}{3} - \boxed{30}\binom{n+1}{2} + \boxed{1}\binom{n}{1} = n^5,$$

$$\boxed{720}\binom{n+5}{6} - \boxed{1800}\binom{n+4}{5} + \boxed{1560}\binom{n+3}{4} - \boxed{540}\binom{n+2}{3} + \boxed{62}\binom{n+1}{2} - \boxed{1}\binom{n}{1} = n^6,$$

$$\vdots \qquad \qquad \ddots$$

On the left-hand side of each line above, do the replacement

$$\begin{array}{ccc} \binom{n}{1} & \text{with} & \binom{n+1}{2}, \\ \binom{n+1}{2} & \text{with} & \binom{n+2}{3}, \\ \binom{n+2}{3} & \text{with} & \binom{n+3}{4}, \\ \binom{n+3}{4} & \text{with} & \binom{n+4}{5}, \end{array}$$

etc. At the same time, on the right-hand side of each line above, do the replacement

$$\begin{array}{lll} n^3 & \text{with} & 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3, \\ n^4 & \text{with} & 1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4, \\ n^5 & \text{with} & 1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5, \\ n^6 & \text{with} & 1^6 + 2^6 + 3^6 + 4^6 + \cdots + n^6, \end{array}$$

etc. Then the resulting identities are valid. Indeed, we have already worked out

$$\begin{array}{l} \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2}, \\ \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n+1}{2} = \binom{n+2}{3}, \\ \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} + \cdots + \binom{n+2}{3} = \binom{n+3}{4}, \\ \binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} + \cdots + \binom{n+3}{4} = \binom{n+4}{5}, \end{array}$$

etc. Therefore the replacement as indicated above amounts to adding up each equation in the previous page side-by-side for

$$n = 1, \quad n = 2, \quad n = 3, \quad n = 4, \quad \cdots, \quad n = n.$$

So let's perform replacement:

$$24\binom{n+4}{5} - 36\binom{n+3}{4} + 14\binom{n+2}{3} - 1\binom{n+1}{2}$$

$$= 1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4.$$

Let's simplify the left-hand side:

$$\begin{aligned} = \quad & 24 \frac{n\left(n+1\right)\left(n+2\right)\left(n+3\right)\left(n+4\right)}{5!} \\ - \; & 36 \frac{n\left(n+1\right)\left(n+2\right)\left(n+3\right)}{4!} \\ + \; & 14 \frac{n\left(n+1\right)\left(n+2\right)}{3!} \\ - \; & 1 \frac{n\left(n+1\right)}{2!} \end{aligned}$$

$$\begin{aligned} = \quad & 24 \frac{n\left(n^4 + 10n^3 + 35n^2 + 50n + 24\right)}{120} \\ - \; & 36 \frac{n\left(n^3 + 6n^2 + 11n + 6\right)}{24} \\ + \; & 14 \frac{n\left(n^2 + 3n + 2\right)}{6} \\ - \; & 1 \frac{n\left(n+1\right)}{2} \end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{5}\Big(n^5 + 10n^4 + 35n^3 + 50n^2 + 24n\Big) \\
& - \frac{3}{2}\Big(n^4 + 6n^3 + 11n^2 + 6n\Big) \\
& + \frac{7}{3}\Big(n^3 + 3n^2 + 2n\Big) \\
& - \frac{1}{2}\Big(n^2 + n\Big)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{30}\Big[\, 6n^5 + 60n^4 + 210n^3 + 300n^2 + 144n \\
& \qquad - 45n^4 - 270n^3 - 495n^2 - 270n \\
& \qquad + 70n^3 + 210n^2 + 140n \\
& \qquad - 15n^2 - 15n \,\Big]
\end{aligned}
$$

$$
= \quad \frac{1}{30}\Big[\, 6n^5 + 15n^4 + 10n^3 - n \,\Big]
$$

$$
= \quad \frac{1}{5}\,n^5 + \frac{1}{2}\,n^4 + \frac{1}{3}\,n^3 - \frac{1}{30}\,n.
$$

This is exactly $1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4$. To summarize:

Formula. Let n be a positive integer. Then

$$
\boxed{1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4 \;=\; \frac{1}{5}\,n^5 + \frac{1}{2}\,n^4 + \frac{1}{3}\,n^3 - \frac{1}{30}\,n} \;.
$$

Similarly, using the numbers in (row 4) of the weighted Pascal in page 5, we have

$$120\binom{n+5}{6} - 240\binom{n+4}{5} + 150\binom{n+3}{4} - 30\binom{n+2}{3} + 1\binom{n+1}{2}$$

$$= 1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5.$$

Let's simplify the left-hand side:

$$\begin{aligned} = \quad & 120 \frac{n(n+1)(n+2)(n+3)(n+4)(n+5)}{6!} \\ - \; & 240 \frac{n(n+1)(n+2)(n+3)(n+4)}{5!} \\ + \; & 150 \frac{n(n+1)(n+2)(n+3)}{4!} \\ - \; & 30 \frac{n(n+1)(n+2)}{3!} \\ + \; & 1 \frac{n(n+1)}{2!} \end{aligned}$$

$$\begin{aligned} = \quad & \frac{1}{6}\left(n^6 + 15n^5 + 85n^4 + 225n^3 + 274n^2 + 120n\right) \\ - \; & 2\left(n^5 + 10n^4 + 35n^3 + 50n^2 + 24n\right) \\ + \; & \frac{25}{4}\left(n^4 + 6n^3 + 11n^2 + 6n\right) \\ - \; & 5\left(n^3 + 3n^2 + 2n\right) \\ + \; & \frac{1}{2}\left(n^2 + n\right) \end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{12}\Big[\, 2n^6 + 30n^5 + 170n^4 + 450n^3 + 548\,n^2 + 240n \\
& \qquad - 24n^5 - 240n^4 - 840n^3 - 1200n^2 - 576n \\
& \qquad + 75\,n^4 + 450n^3 + 825\,n^2 + 450n \\
& \qquad - 60\,n^3 - 180\,n^2 - 120n \\
& \qquad + 6\,n^2 + 6\,n \,\Big]
\end{aligned}
$$

$$
= \quad \frac{1}{12}\Big[\, 2n^6 + 6\,n^5 + 5\,n^4 - n^2 \,\Big]
$$

$$
= \quad \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2.
$$

This is exactly $1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5.$ To summarize:

Formula. Let n be a positive integer. Then

$$
\boxed{1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2}\,.
$$

We may employ the same strategy to pull formula for

$$
1^6 + 2^6 + 3^6 + 4^6 + \cdots + n^6.
$$

Namely, using the numbers in (row 5) of the weighted Pascal in page 5, we have

$$720\binom{n+6}{7} - 1800\binom{n+5}{6} + 1560\binom{n+4}{5} - 540\binom{n+3}{4} + 62\binom{n+2}{3} - 1\binom{n+1}{2}$$

$$= \; 1^6 + 2^6 + 3^6 + 4^6 + \cdots + n^6.$$

By simplifying the left-hand side, we obtain:

Formula. Let n be a positive integer. Then

$$\boxed{1^6 + 2^6 + 3^6 + 4^6 + \cdots + n^6 \; = \; \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n}\,.$$

"In theory", this strategy works for any higher integer powers, and we can get formulas for

$$1^7 + 2^7 + 3^7 + 4^7 + \cdots + n^7,$$

$$1^8 + 2^8 + 3^8 + 4^8 + \cdots + n^8,$$

$$1^9 + 2^9 + 3^9 + 4^9 + \cdots + n^9,$$

and so on.

Now, there is an alternative startegy:

Strategy B. Rely on the diagram in the next page. This is actually the same diagram we briefly touched in "Review of Lectures – XXI". At that time we threw a concrete number for n, but the algorithm can be performed for all n at once. The diagram in the next page is the demonstration of that algorithm for all n at once. In that case the entries are all polynomials in n. This diagram is pretty self-explanatory. The top row is just the binomial coefficients, and the left-most in each row yields the formulas for

$$1^1 + 2^1 + 3^1 + 4^1 + \cdots + n^1,$$

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2,$$

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3,$$

$$1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4,$$

and so on.

$$\binom{n+1}{2} = \frac{n(n+1)}{2!} = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\binom{n+2}{3} = \frac{n(n+1)(n+2)}{3!} = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$$

$$\binom{n+3}{4} = \frac{n(n+1)(n+2)(n+3)}{4!} = \frac{1}{24}n^4 + \frac{1}{4}n^3 + \frac{11}{24}n^2 + \frac{1}{4}n$$

$$\binom{n+4}{4} = \frac{n(n+1)(n+2)(n+3)(n+4)}{5!} = \frac{1}{120}n^5 + \frac{1}{12}n^4 + \frac{7}{24}n^3 + \frac{5}{12}n^2 + \frac{1}{5}n$$

(row 1)

$n{+}1 \backslash$ $/{-1}$ $n{+}2 \backslash$ $/{-2}$ $n{+}3 \backslash$ $/{-3}$ $n{+}4 \backslash$

$$\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \qquad \frac{1}{12}n^4 + \frac{1}{3}n^3 + \frac{5}{12}n^2 + \frac{1}{6}n \qquad \frac{1}{60}n^5 + \frac{1}{8}n^4 + \frac{1}{3}n^3 + \frac{3}{8}n^2 + \frac{3}{20}n$$

(row 2)

$n{+}1 \backslash$ $/{-1}$ $n{+}2 \backslash$ $/{-2}$ $n{+}3 \backslash$

$$\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \qquad \frac{1}{20}n^5 + \frac{1}{4}n^4 + \frac{5}{12}n^3 + \frac{1}{4}n^2 + \frac{1}{30}n$$

(row 3)

$n{+}1 \backslash$ $/{-1}$ $n{+}2 \backslash$

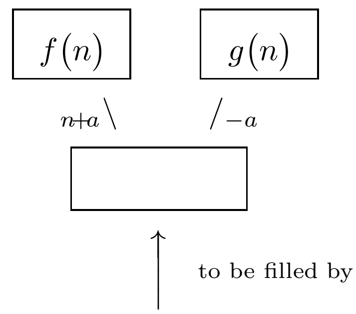
$$\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

(row 4)

$n{+}1 \backslash$

The diagram can be endlessly stretched towards the right, where the top row preserves the same, self-evident, pattern. You go down row by row. At each step, you obey the rule:

Rule:



$$\left(n+a\right) \cdot f\left(n\right) \; - \; a \cdot g\left(n\right).$$

Then the formula for

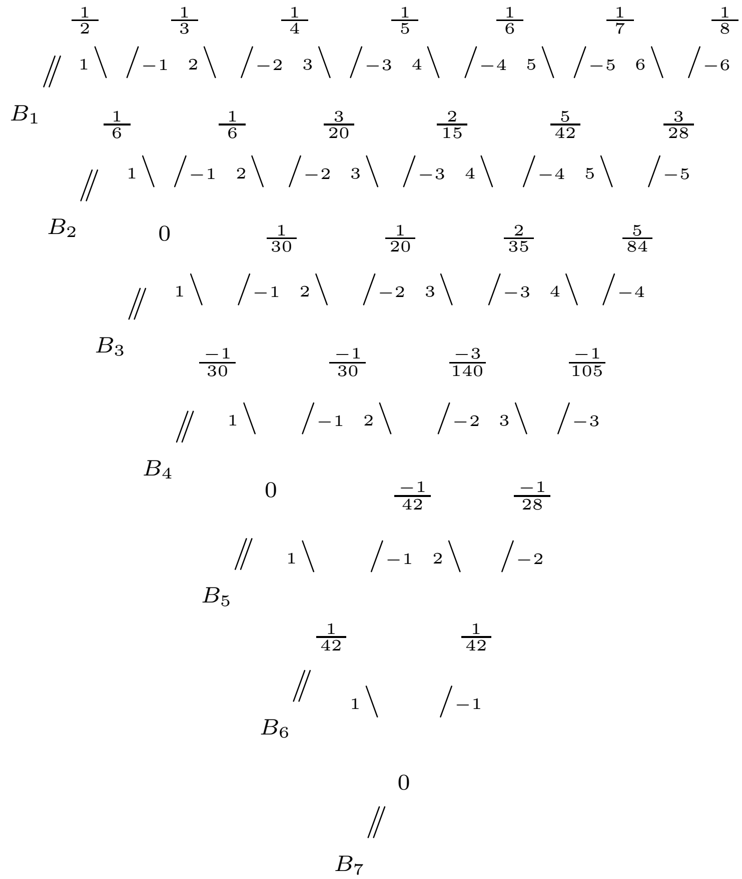
$$1^k + 2^k + 3^k + 4^k + \cdots + n^k$$

is the left-most in $\left(\text{row } k\right)$.

You need to write out up to $\binom{n+k}{k}$ in the top row in order to attain it.

- Now, there is actually a thrid strategy. That is, instead of going on with the entire polynomials in the porevious diagram, you pick out the coefficient of n.

Strategy C. Work with the following alternative diagram consisting of numbers, instead of polynomials:



This is called “the Akiyama–Tanigawa diagram”.* Just like the last one, this diagram can be endlessly stretched towards the right, where the top row preserves the same, self-evident, pattern. You go down row by row. How to recover the formulas for

$$1^k + 2^k + 3^k + 4^k + \cdots + n^k$$

is as follows:

First, binomially expand

$$\frac{\left(n+B\right)^2 - B^2}{2}, \quad \frac{\left(n+B\right)^3 - B^3}{3}, \quad \frac{\left(n+B\right)^4 - B^4}{4}, \quad \frac{\left(n+B\right)^5 - B^5}{5}, \quad \cdots,$$

as in

*Discovered by Shigeki Akiyama and Yoshio Tanigawa in 1997.

$$\frac{\left(n+B\right)^2 - B^1}{2} = \frac{\binom{2}{0}}{2} n^2 + \frac{\binom{2}{1} B^2}{2} n,$$

$$\frac{\left(n+B\right)^3 - B^3}{3} = \frac{\binom{3}{0}}{3} n^3 + \frac{\binom{3}{1} B^1}{3} n^2 + \frac{\binom{3}{2} B^2}{3} n,$$

$$\frac{\left(n+B\right)^4 - B^4}{4} = \frac{\binom{4}{0}}{4} n^4 + \frac{\binom{4}{1} B^1}{4} n^3 + \frac{\binom{4}{2} B^2}{4} n^2 + \frac{\binom{4}{3} B^3}{4} n,$$

$$\frac{\left(n+B\right)^5 - B^5}{5} = \frac{\binom{5}{0}}{5} n^5 + \frac{\binom{5}{1} B^1}{5} n^4 + \frac{\binom{5}{2} B^2}{5} n^3 + \frac{\binom{5}{3} B^3}{5} n^2 + \frac{\binom{5}{4} B^4}{5} n,$$

$$\vdots \qquad \qquad \ddots$$

Next, 'lower' the exponents for B, namely, replace B^1 with B_1; B^2 with B_2; B^3 with B_3, and so on:

$$\frac{\binom{2}{0}}{2} n^2 + \frac{\binom{2}{1} B_2}{2} n,$$

$$\frac{\binom{3}{0}}{3} n^3 + \frac{\binom{3}{1} B_1}{3} n^2 + \frac{\binom{3}{2} B_2}{3} n,$$

$$\frac{\binom{4}{0}}{4} n^4 + \frac{\binom{4}{1} B_1}{4} n^3 + \frac{\binom{4}{2} B_2}{4} n^2 + \frac{\binom{4}{3} B_3}{4} n,$$

$$\frac{\binom{5}{0}}{5} n^5 + \frac{\binom{5}{1} B_1}{5} n^4 + \frac{\binom{5}{2} B_2}{5} n^3 + \frac{\binom{5}{3} B_3}{5} n^2 + \frac{\binom{5}{4} B_4}{5} n,$$

$$\vdots \qquad \qquad \ddots$$

Last, substitute $B_1,\ B_2,\ B_3,\ B_4,\ B_5,\ \cdots$ from the previous diagram. So

$$\begin{array}{l}
\frac{\binom{2}{0}}{2}n^2 + \frac{\binom{2}{1}}{2}\frac{1}{2}n, \\[2ex]
\frac{\binom{3}{0}}{3}n^3 + \frac{\binom{3}{1}}{3}\frac{1}{2}n^2 + \frac{\binom{3}{2}}{3}\frac{1}{6}n, \\[2ex]
\frac{\binom{4}{0}}{4}n^4 + \frac{\binom{4}{1}}{4}\frac{1}{2}n^3 + \frac{\binom{4}{2}}{4}\frac{1}{6}n^2, \\[2ex]
\frac{\binom{5}{0}}{5}n^5 + \frac{\binom{5}{1}}{5}\frac{1}{2}n^4 + \frac{\binom{5}{2}}{5}\frac{1}{6}n^3 + \frac{\binom{5}{4}}{5}\frac{-1}{30}n, \\
\quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddots
\end{array}$$

These are simplified as

$$\begin{array}{l}
\frac{1}{2}n^2 + \frac{1}{2}n, \\[2ex]
\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n, \\[2ex]
\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2, \\[2ex]
\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n, \\
\quad\vdots
\end{array}$$

These are nothing else but the formulas for

$$1^1 + 2^1 + 3^1 + 4^1 + \cdots + n^1,$$
$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2,$$
$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3,$$
$$1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4,$$
$$\vdots$$

Now, the numbers $B_1,\ B_2,\ B_3,\ \cdots$ are extremely important, important enough that they have a name. They are called the Bernoulli numbers . Bernoulli numbers were discovered in the early 18th century, independently by two mathematicians, called Jakob Bernoulli and Kowa Seki.* Bernoulli numbers are all rational numbers. The first few of them:

$$\begin{aligned}
B_1 &= \frac{1}{2}, & B_2 &= \frac{1}{6}, \\
B_3 &= 0, & B_4 &= \frac{-1}{30}, \\
B_5 &= 0, & B_6 &= \frac{1}{42}, \\
B_7 &= 0, & B_8 &= \frac{-1}{30}, \\
B_9 &= 0, & B_{10} &= \frac{5}{66}, \\
B_{11} &= 0, & B_{12} &= \frac{-691}{2730} \\
& & & \left(= 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{13}\right).^{**} \\
&\vdots
\end{aligned}$$

Note that 691 is a prime. With the exception of B_1, all the Bernoulli numbers with odd suffixes are are 0.

*Jakob Bernoulli (1654–1705), Kowa (= Takakazu) Seki (1642(?)–1708).

**See "*The Book of Numbers*" by John H. Conway and Richard K. Guy, Copernicus, Springer Science+Business Media, Inc. 2006, page 106–109.