Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XXV

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§25. Derivatives of polynomials.

We have learned that addition, subtraction and multiplication of polynomials. There is another important operation involving polynomials, called <u>the derivative</u>. Spelling:

"derivative."

Caution (!) You hear this word 'derivative' in finance. That's completely different. This is not a finance class. This is math. What I am going to talk about is 'derivative' as a mathematical notion, which was discovered independently by Newton and Leibniz.*

⋆ Another lingo: The following words refer to the action 'to take the derivative (of something)':

"differentiation" (n), "differentiate" (v).

So,

" differentiate $f(x)$ " and " take (find) the derivative of $f(x)$"

are completely synonymous.

Today we focus on differentiating polynomials. First, notation:

*Isaac Newton (1643–1727), Gottfried Leibniz (1646–1716).

Notation. The derivative of $f\left(x\right)$ is denoted as

$$\frac{d}{dx}\,f\left(x\right), \qquad \frac{df}{dx}, \qquad \frac{df\left(x\right)}{dx}, \qquad \text{or} \qquad f'\left(x\right).$$

Suppose $f\left(x\right) = 3x^2 + 4x + 4$. Then you can write its derivative as either

$$\frac{d}{dx}\,f\left(x\right), \qquad \frac{d}{dx}\left(3x^2 + 4x + 4\right), \qquad \text{or} \qquad f'\left(x\right).$$

On the other hand,

$$\left(3x^2 + 4x + 4\right)'$$

is undesirable, so avoid this way of writing. Below is the first thing you need to know:

First rule.

$$\begin{aligned} \frac{d}{dx}\,1 &= 0, \\ \frac{d}{dx}\,x &= 1, \\ \frac{d}{dx}\,x^2 &= 2x, \\ \frac{d}{dx}\,x^3 &= 3x^2, \\ \frac{d}{dx}\,x^4 &= 4x^3, \\ \frac{d}{dx}\,x^5 &= 5x^4, \\ \frac{d}{dx}\,x^6 &= 6x^5, \\ &\vdots \end{aligned}$$

More generally:

Definition. Let n be an integer, $n \geq 0$. Then

$$\boxed{\frac{d}{dx}\,x^n \;=\; n\,x^{n-1}}\,.$$

Next, how you handle constant multiplication inside $\dfrac{d}{dx}$.

Example 1. $\dfrac{d}{dx}\left(9x^4\right)$. This goes as follows:

$$\boxed{\frac{d}{dx}}\left(\boxed{9}\;x^4\right) \;=\; \boxed{9}\;\cdot\;\boxed{\frac{d}{dx}}\;x^4$$

swap (swapped)

$$= \;\; 9 \cdot 4 \cdot x^3$$

$$= \;\; 36 \cdot x^3.$$

Example 2. $\dfrac{d}{dx}\left(7x^6\right)$. This goes as follows:

$$\boxed{\frac{d}{dx}}\left(\boxed{7}\;x^6\right) \;=\; \boxed{7}\;\cdot\;\boxed{\frac{d}{dx}}\;x^6$$

swap (swapped)

$$= \;\; 7 \cdot 6 \cdot x^5$$

$$= \;\; 42 \cdot x^5.$$

Example 3. $\dfrac{d}{dx}\Big(20x^5\Big).$ This goes as follows:

$$\boxed{\frac{d}{dx}}\left(\boxed{20}\;x^5\right) \;=\; \boxed{20}\cdot\boxed{\frac{d}{dx}}\;x^5$$

swap (swapped)

$$= \;\; 20 \cdot 5 \cdot x^4$$

$$= \;\; 100 \cdot x^4.$$

⋆ In short, the rule is, whenever you see a constant inside $\dfrac{d}{dx}$, drag it outside $\dfrac{d}{dx}$. You may write the above three results as

$$\frac{d}{dx}\Big(9x^4\Big) \;=\; 9\cdot 4\cdot x^3 \;=\; 36\cdot x^3,$$

$$\frac{d}{dx}\Big(14x^6\Big) \;=\; 14\cdot 6\cdot x^5 \;=\; 84\cdot x^5,$$

$$\frac{d}{dx}\Big(20x^5\Big) \;=\; 20\cdot 5\cdot x^4 \;=\; 100\cdot x^4.$$

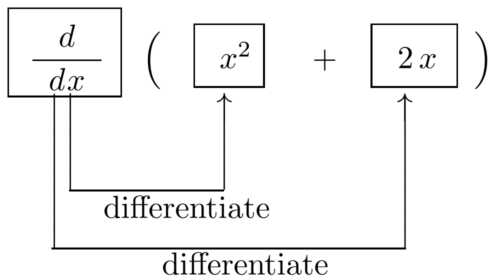
and so on. The following definition makes it official:

Definition. Let a be a constant real number. Let n be an integer, $n \geq 0$. Then

$$\boxed{\frac{d}{dx}\Big(ax^n\Big) \;=\; a\,\frac{d}{dx}\,x^n \;=\; a\,n\,x^{n-1}}\;.$$

$\star$ Finally, $\dfrac{d}{dx}$ breaks up addition.

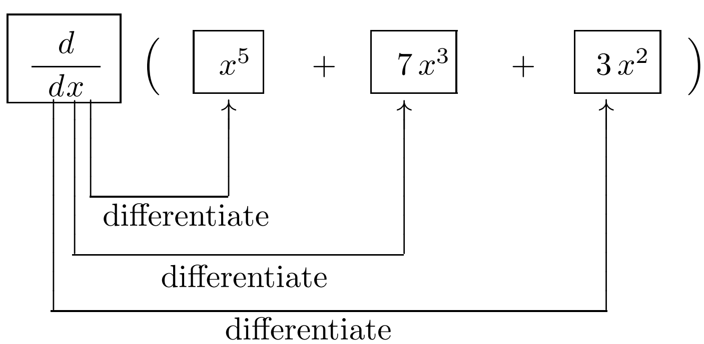
Example 4. $\dfrac{d}{dx}\left(x^2 + 2x\right)$. This goes as follows:



$$= \left[\frac{d}{dx}\,x^2\right] + \left[\frac{d}{dx}\left(2x\right)\right]$$

$$= 2x + 2.$$

Example 5. $\dfrac{d}{dx}\left(x^5 + 7x^3 + 3x^2\right)$. This goes as follows:



$$= \left[\frac{d}{dx}\,x^5\right] + \left[\frac{d}{dx}\left(7x^3\right)\right] + \left[\frac{d}{dx}\left(3x^2\right)\right]$$

$$= 5x^4 + 7\cdot 3x^2 + 3\cdot 2x = 5x^4 + 21x^2 + 6x.$$

The following definition makes it official:

Definition. Let $f(x)$ and $g(x)$ be monomials. Then

$$\boxed{\frac{d}{dx}\Big(f(x) + g(x)\Big) = f'(x) + g'(x)}\ .$$

$$\boxed{\frac{d}{dx}\Big(f(x) + g(x) + h(x)\Big) = f'(x) + g'(x) + h'(x)}\ .$$

⋆ When you are to to differentiate a polynomial consisting of four or more monomials, the same thing.

Now, the above allows you to in theory calculate derivative of polynomial.

Example 6. Let's differentiate

$$f(x) = x^2 + 2x + 5.$$

It goes as follows:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\Big(x^2 + 2x + 5\Big) \\ &= \left[\frac{d}{dx}x^2\right] + \left[\frac{d}{dx}\Big(2x\Big)\right] + \left[\frac{d}{dx}5\right] \\ &= 2x + 2 + 0 \\ &= 2x + 2. \end{aligned}$$

Note that this answer is the same as Example 4 $\big($page 5$\big)$. Notice that the polynomial before differentiating in Example 4 and the polynomial before differentiating in Example 6 differ only by a constant. Then the resulting derivatives will coincide.

Example 7. Let's differentiate

$$f(x) = 4x^4 + 8x^3 - 7x^2.$$

It goes as follows:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(4x^4 + 8x^3 - 7x^2\right) \\ &= \left[\frac{d}{dx}\left(4x^4\right)\right] + \left[\frac{d}{dx}\left(8x^3\right)\right] - \left[\frac{d}{dx}\left(7x^2\right)\right] \\ &= 4 \cdot 4x^3 + 8 \cdot 3x^2 - 7 \cdot 2x \\ &= 16x^3 + 24x^2 - 14x. \end{aligned}$$

Exercise 1. Do the following differentiation:

(1) $\dfrac{d}{dx}\, x^{10}.$ (2) $\dfrac{d}{dx}\, 2x^8.$ (3) $\dfrac{d}{dx}\, 4x^{50}.$

(4) $\dfrac{d}{dx}\, x^{1000}.$

[**Answers**]:

(1) $10x^9.$ (2) $16x^7.$

(3) $200x^{49}.$ (4) $1000x^{999}.$

Exercise 2. Do the following differentiation:

(1) $\dfrac{d}{dx}\Big(2x^3 - 11x^2 + 4x\Big).$

(2) $\dfrac{d}{dx}\Big(8x^8 - 12x^6 + 24x^4 - 64x^2 + 96\Big).$

(3) $\dfrac{d}{dx}\left(\dfrac{1}{4}x^4 - \dfrac{1}{2}x^3 + \dfrac{1}{6}x^2 - \dfrac{1}{30}\right).$

(4) $\dfrac{d}{dx}\left(\dfrac{10}{7}x^7 + \dfrac{3}{2}x^6 + 3x^5\right).$

[**Answers**]**:**

(1) $6x^2 - 22x + 4.$ (2) $64x^7 - 72x^5 + 96x^3 - 128x.$

(3) $x^3 - \dfrac{3}{2}x^2 + \dfrac{1}{3}x.$ (4) $10x^6 + 9x^5 + 15x^4.$

Exercise 3. Do the following differentiation:

(1) $$\frac{d}{dx}\left(\frac{1}{1!}\,x\right).$$

(2) $$\frac{d}{dx}\left(\frac{1}{2!}\,x^2\right).$$

(3) $$\frac{d}{dx}\left(\frac{1}{3!}\,x^3\right).$$

(4) $$\frac{d}{dx}\left(\frac{1}{4!}\,x^4\right).$$

(5) $$\frac{d}{dx}\left(\frac{1}{5!}\,x^5\right).$$

(5) $$\frac{d}{dx}\left(\frac{1}{6!}\,x^6\right).$$

Answers:

(1) $$\frac{d}{dx}\left(\frac{1}{1!}\,x\right) \quad = \quad 1.$$

(2) $$\frac{d}{dx}\left(\frac{1}{2!}\,x^2\right) \quad = \quad \frac{1}{1!}\,x \quad \left(= \quad x\right).$$

(3) $$\frac{d}{dx}\left(\frac{1}{3!}\,x^3\right) \quad = \quad \frac{1}{2!}\,x^2.$$

(4) $$\frac{d}{dx}\left(\frac{1}{4!}\,x^4\right) \quad = \quad \frac{1}{3!}\,x^3.$$

(5) $$\frac{d}{dx}\left(\frac{1}{5!}\,x^5\right) \quad = \quad \frac{1}{4!}\,x^4.$$

(6) $$\frac{d}{dx}\left(\frac{1}{6!}\,x^6\right) \quad = \quad \frac{1}{5!}\,x^5.$$

Pop Quiz. $$\frac{d}{dx}\left(\frac{1}{20!}\,x^{20}\right) \quad =?$$

Answer: $$\frac{1}{19!}\,x^{19}.$$

Pop Quiz. $$\frac{d}{dx}\left(\frac{1}{100!}\,x^{100}\right) \quad =?$$

Answer: $$\frac{1}{99!}\,x^{99}.$$

Pop Quiz. Do the following differentiation:

$$\frac{d}{dx}\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8\right).$$

[**Answer**]: $1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7.$

Pop Quiz. Do the following differentiation:

$$\frac{d}{dx}\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8+\frac{1}{9!}x^9\right.$$
$$+\frac{1}{10!}x^{10}+\frac{1}{11!}x^{11}+\frac{1}{12!}x^{12}+\frac{1}{13!}x^{13}+\frac{1}{14!}x^{14}+\frac{1}{15!}x^{15}+\frac{1}{16!}x^{16}+\frac{1}{17!}x^{17}+\frac{1}{18!}x^{18}+\frac{1}{19!}x^{19}$$
$$+\frac{1}{20!}x^{20}+\frac{1}{21!}x^{21}+\frac{1}{22!}x^{22}+\frac{1}{23!}x^{23}+\frac{1}{24!}x^{24}+\frac{1}{25!}x^{25}+\frac{1}{26!}x^{26}+\frac{1}{27!}x^{27}+\frac{1}{28!}x^{28}+\frac{1}{29!}x^{29}$$
$$+\frac{1}{30!}x^{30}+\frac{1}{31!}x^{31}+\frac{1}{32!}x^{32}+\frac{1}{33!}x^{33}+\frac{1}{34!}x^{34}+\frac{1}{35!}x^{35}+\frac{1}{36!}x^{36}+\frac{1}{37!}x^{37}+\frac{1}{38!}x^{38}+\frac{1}{39!}x^{39}$$
$$\left.+\frac{1}{40!}x^{40}+\frac{1}{41!}x^{41}+\frac{1}{42!}x^{42}+\frac{1}{43!}x^{43}+\frac{1}{44!}x^{44}+\frac{1}{45!}x^{45}+\frac{1}{46!}x^{46}+\frac{1}{47!}x^{47}+\frac{1}{48!}x^{48}+\frac{1}{49!}x^{49}\right).$$

[**Answer**]:

$$1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8+\frac{1}{9!}x^9$$
$$+\frac{1}{10!}x^{10}+\frac{1}{11!}x^{11}+\frac{1}{12!}x^{12}+\frac{1}{13!}x^{13}+\frac{1}{14!}x^{14}+\frac{1}{15!}x^{15}+\frac{1}{16!}x^{16}+\frac{1}{17!}x^{17}+\frac{1}{18!}x^{18}+\frac{1}{19!}x^{19}$$
$$+\frac{1}{20!}x^{20}+\frac{1}{21!}x^{21}+\frac{1}{22!}x^{22}+\frac{1}{23!}x^{23}+\frac{1}{24!}x^{24}+\frac{1}{25!}x^{25}+\frac{1}{26!}x^{26}+\frac{1}{27!}x^{27}+\frac{1}{28!}x^{28}+\frac{1}{29!}x^{29}$$
$$+\frac{1}{30!}x^{30}+\frac{1}{31!}x^{31}+\frac{1}{32!}x^{32}+\frac{1}{33!}x^{33}+\frac{1}{34!}x^{34}+\frac{1}{35!}x^{35}+\frac{1}{36!}x^{36}+\frac{1}{37!}x^{37}+\frac{1}{38!}x^{38}+\frac{1}{39!}x^{39}$$
$$+\frac{1}{40!}x^{40}+\frac{1}{41!}x^{41}+\frac{1}{42!}x^{42}+\frac{1}{43!}x^{43}+\frac{1}{44!}x^{44}+\frac{1}{45!}x^{45}+\frac{1}{46!}x^{46}+\frac{1}{47!}x^{47}+\frac{1}{48!}x^{48}.$$

Pop Quiz. Do the following differentiation:

$$\frac{d}{dx}\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8+\frac{1}{9!}x^9\right.$$

$$+\frac{1}{10!}x^{10}+\frac{1}{11!}x^{11}+\frac{1}{12!}x^{12}+\frac{1}{13!}x^{13}+\frac{1}{14!}x^{14}+\frac{1}{15!}x^{15}+\frac{1}{16!}x^{16}+\frac{1}{17!}x^{17}+\frac{1}{18!}x^{18}+\frac{1}{19!}x^{19}$$

$$+\frac{1}{20!}x^{20}+\frac{1}{21!}x^{21}+\frac{1}{22!}x^{22}+\frac{1}{23!}x^{23}+\frac{1}{24!}x^{24}+\frac{1}{25!}x^{25}+\frac{1}{26!}x^{26}+\frac{1}{27!}x^{27}+\frac{1}{28!}x^{28}+\frac{1}{29!}x^{29}$$

$$+\frac{1}{30!}x^{30}+\frac{1}{31!}x^{31}+\frac{1}{32!}x^{32}+\frac{1}{33!}x^{33}+\frac{1}{34!}x^{34}+\frac{1}{35!}x^{35}+\frac{1}{36!}x^{36}+\frac{1}{37!}x^{37}+\frac{1}{38!}x^{38}+\frac{1}{39!}x^{39}$$

$$+\frac{1}{40!}x^{40}+\frac{1}{41!}x^{41}+\frac{1}{42!}x^{42}+\frac{1}{43!}x^{43}+\frac{1}{44!}x^{44}+\frac{1}{45!}x^{45}+\frac{1}{46!}x^{46}+\frac{1}{47!}x^{47}+\frac{1}{48!}x^{48}+\frac{1}{49!}x^{49}$$

$$+\frac{1}{50!}x^{50}+\frac{1}{51!}x^{51}+\frac{1}{52!}x^{52}+\frac{1}{53!}x^{53}+\frac{1}{54!}x^{54}+\frac{1}{55!}x^{55}+\frac{1}{56!}x^{56}+\frac{1}{57!}x^{57}+\frac{1}{58!}x^{58}+\frac{1}{59!}x^{59}$$

$$+\frac{1}{60!}x^{60}+\frac{1}{61!}x^{61}+\frac{1}{62!}x^{62}+\frac{1}{63!}x^{63}+\frac{1}{64!}x^{64}+\frac{1}{65!}x^{65}+\frac{1}{66!}x^{66}+\frac{1}{67!}x^{67}+\frac{1}{68!}x^{68}+\frac{1}{69!}x^{69}$$

$$+\frac{1}{70!}x^{70}+\frac{1}{71!}x^{71}+\frac{1}{72!}x^{72}+\frac{1}{73!}x^{73}+\frac{1}{74!}x^{74}+\frac{1}{75!}x^{75}+\frac{1}{76!}x^{76}+\frac{1}{77!}x^{77}+\frac{1}{78!}x^{78}+\frac{1}{79!}x^{79}$$

$$+\frac{1}{80!}x^{80}+\frac{1}{81!}x^{81}+\frac{1}{82!}x^{82}+\frac{1}{83!}x^{83}+\frac{1}{84!}x^{84}+\frac{1}{85!}x^{85}+\frac{1}{86!}x^{86}+\frac{1}{87!}x^{87}+\frac{1}{88!}x^{88}+\frac{1}{89!}x^{89}$$

$$+\frac{1}{90!}x^{90}+\frac{1}{91!}x^{91}+\frac{1}{92!}x^{92}+\frac{1}{93!}x^{93}+\frac{1}{94!}x^{94}+\frac{1}{95!}x^{95}+\frac{1}{96!}x^{96}+\frac{1}{97!}x^{97}+\frac{1}{98!}x^{98}+\frac{1}{99!}x^{99}$$

$$\left.+\frac{1}{100!}x^{100}\right).$$

Answer:

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \frac{1}{9!}x^9$$

$$+\frac{1}{10!}x^{10}+\frac{1}{11!}x^{11}+\frac{1}{12!}x^{12}+\frac{1}{13!}x^{13}+\frac{1}{14!}x^{14}+\frac{1}{15!}x^{15}+\frac{1}{16!}x^{16}+\frac{1}{17!}x^{17}+\frac{1}{18!}x^{18}+\frac{1}{19!}x^{19}$$

$$+\frac{1}{20!}x^{20}+\frac{1}{21!}x^{21}+\frac{1}{22!}x^{22}+\frac{1}{23!}x^{23}+\frac{1}{24!}x^{24}+\frac{1}{25!}x^{25}+\frac{1}{26!}x^{26}+\frac{1}{27!}x^{27}+\frac{1}{28!}x^{28}+\frac{1}{29!}x^{29}$$

$$+\frac{1}{30!}x^{30}+\frac{1}{31!}x^{31}+\frac{1}{32!}x^{32}+\frac{1}{33!}x^{33}+\frac{1}{34!}x^{34}+\frac{1}{35!}x^{35}+\frac{1}{36!}x^{36}+\frac{1}{37!}x^{37}+\frac{1}{38!}x^{38}+\frac{1}{39!}x^{39}$$

$$+\frac{1}{40!}x^{40}+\frac{1}{41!}x^{41}+\frac{1}{42!}x^{42}+\frac{1}{43!}x^{43}+\frac{1}{44!}x^{44}+\frac{1}{45!}x^{45}+\frac{1}{46!}x^{46}+\frac{1}{47!}x^{47}+\frac{1}{48!}x^{48}+\frac{1}{49!}x^{49}$$

$$+\frac{1}{50!}x^{50}+\frac{1}{51!}x^{51}+\frac{1}{52!}x^{52}+\frac{1}{53!}x^{53}+\frac{1}{54!}x^{54}+\frac{1}{55!}x^{55}+\frac{1}{56!}x^{56}+\frac{1}{57!}x^{57}+\frac{1}{58!}x^{58}+\frac{1}{59!}x^{59}$$

$$+\frac{1}{60!}x^{60}+\frac{1}{61!}x^{61}+\frac{1}{62!}x^{62}+\frac{1}{63!}x^{63}+\frac{1}{64!}x^{64}+\frac{1}{65!}x^{65}+\frac{1}{66!}x^{66}+\frac{1}{67!}x^{67}+\frac{1}{68!}x^{68}+\frac{1}{69!}x^{69}$$

$$+\frac{1}{70!}x^{70}+\frac{1}{71!}x^{71}+\frac{1}{72!}x^{72}+\frac{1}{73!}x^{73}+\frac{1}{74!}x^{74}+\frac{1}{75!}x^{75}+\frac{1}{76!}x^{76}+\frac{1}{77!}x^{77}+\frac{1}{78!}x^{78}+\frac{1}{79!}x^{79}$$

$$+\frac{1}{80!}x^{80}+\frac{1}{81!}x^{81}+\frac{1}{82!}x^{82}+\frac{1}{83!}x^{83}+\frac{1}{84!}x^{84}+\frac{1}{85!}x^{85}+\frac{1}{86!}x^{86}+\frac{1}{87!}x^{87}+\frac{1}{88!}x^{88}+\frac{1}{89!}x^{89}$$

$$+\frac{1}{90!}x^{90}+\frac{1}{91!}x^{91}+\frac{1}{92!}x^{92}+\frac{1}{93!}x^{93}+\frac{1}{94!}x^{94}+\frac{1}{95!}x^{95}+\frac{1}{96!}x^{96}+\frac{1}{97!}x^{97}+\frac{1}{98!}x^{98}+\frac{1}{99!}x^{99}.$$