Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – XXIV

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§24. Polynomials and their arithmetic – III.

• Squaring Polynomials.

The next item is squaring polynomials. Let's remember that squaring of something means multiply the two identical copies of that thing, like $2^2 = 2 \cdot 2$, $5^2 = 5 \cdot 5$, *etc.* The same for polynomials. If $f\left(x\right)$ is a polynomial, then $f\left(x\right)^2$ means just $f\left(x\right) \cdot f\left(x\right)$.

Example 1. Let's expand

$$\left(x^2 + 3\right)^2.$$

This is the same as

$$\left(x^2 + 3\right)\left(x^2 + 3\right).$$

So

$$\begin{aligned}\left(x^2 + 3\right)\left(x^2 + 3\right) &= x^2\left(x^2 + 3\right) + 3\left(x^2 + 3\right) \\ &= x^4 + 3\,x^2 + 3\,x^2 + 9 \\ &= x^4 + 6\,x^2 + 9.\end{aligned}$$

Now, some of you might say "why not use the binomial formula?". That's excellent point. Yes, let's recall

$$\left(a + b\right)^2 = a^2 + 2ab + b^2.$$

So

$$\begin{aligned}\left(a + 3\right)^2 &= a^2 + 2 \cdot 3 \cdot a + 3^2 \\ &= a^2 + 6\,a + 9.\end{aligned}$$

Substitute a with x^2:

$$\begin{aligned}\left(x^2 + 3\right)^2 &= \left(x^2\right)^2 + 6\,x^2 + 9 \\ &= x^4 + 6\,x^2 + 9.\end{aligned}$$

We certainly get the same answer. Now, relying on the binomial formula was feasible because the original polynomial consisted of two terms. How about the following:

Example 2. Let's expand

$$\left(x^2 + x + 2\right)^2.$$

This time you have to do it 'honestly', like

$$\left(x^2 + x + 2\right)\left(x^2 + x + 2\right).$$

It goes as follows:

$$\begin{aligned}&\left(x^2 + x + 2\right)\left(x^2 + x + 2\right) \\ &\quad = x^2\left(x^2 + x + 2\right) + x\left(x^2 + x + 2\right) + 2\left(x^2 + x + 2\right) \\ &\quad = \left(x^4 + x^3 + 2x^2\right) + \left(x^3 + x^2 + 2x\right) + \left(2x^2 + 2x + 4\right)\end{aligned}$$

$$
\begin{aligned}
&= x^4 + x^3 + 2x^2 + x^3 + x^2 + 2x + 2x^2 + 2x + 4 \\
&= x^4 + x^3 + x^3 + 2x^2 + x^2 + 2x^2 + 2x + 2x + 4 \\
&= x^4 + 2x^3 + 5x^2 + 4x + 4.
\end{aligned}
$$

You can certainly do it this way:

$$
\begin{array}{rrrrrrrrrr}
 & & & & & x^2 & + & x & + & 2 \\
 & & & & & x^2 & + & x & + & 2 \\
\times) & & & & & & & & & \\
\hline
 & & & & & 2\,x^2 & + & 2\,x & + & 4 \\
 & & & x^3 & + & x^2 & + & 2\,x & & \\
 & x^4 & + & x^3 & + & 2\,x^2 & & & & \\
\hline
 & x^4 & + & 2\,x^3 & + & 5\,x^2 & + & 4\,x & + & 4
\end{array}
$$

To conclude,

$$\left(x^2 + x + 2\right)^2 = x^4 + 2x^3 + 5x^2 + 4x + 4.$$

You can do it either way. Let's do more examples.

Example 3. Let's expand

$$\left(x^3 + 4x^2 - 3x + 2\right)^2.$$

Let's just do it in the second way.

$$
\begin{array}{rrrrrrrr}
 & & & & x^3 & +\ 4\,x^2 & -\ 3\,x & +\ 2 \\
 & & & & x^3 & +\ 4\,x^2 & -\ 3\,x & +\ 2 \\
\hline
 & & & & 2\,x^3 & +\ 8\,x^2 & -\ 6\,x & +\ 4 \\
 & & & -\ 3\,x^4 & -\ 12\,x^3 & +\ 9\,x^2 & -\ 6\,x & \\
 & & 4\,x^5 & +\ 16\,x^4 & -\ 12\,x^3 & +\ 8\,x^2 & & \\
 & x^6 & +\ 4\,x^5 & -\ 3\,x^4 & +\ 2\,x^3 & & & \\
\hline
 & x^6 & +\ 8\,x^5 & +\ 10\,x^4 & -\ 20\,x^3 & +\ 25\,x^2 & -\ 12\,x & +\ 4
\end{array}
$$

$\times)$

To conclude,

$$\left(x^3 + 4x^2 - 3x + 2\right)^2 = x^6 + 8x^5 + 10x^4 - 20x^3 + 25x^2 - 12x + 4.$$

Example 4. Let's expand $\left(x^4 - 5x^3 - 6\,x + 4\right)^2$.

As before, we can handle it like

$$
\begin{array}{rrrrrrrrr}
 & & & & x^4 & -\ 5\,x^3 & & -\ 6\,x & +\ 4 \\
 & & & & x^4 & -\ 5\,x^3 & & -\ 6\,x & +\ 4 \\
\hline
 & & & & 4\,x^4 & -\ 20\,x^3 & & -\ 24\,x & +\ 16 \\
 & & & -\ 6\,x^5 & +\ 30\,x^4 & & +\ 36\,x^2 & -\ 24\,x & \\
 & -\ 5\,x^7 & +\ 25\,x^6 & & +\ 30\,x^4 & -\ 20\,x^3 & & & \\
x^8 & -\ 5\,x^7 & & -\ 6\,x^5 & +\ 4\,x^4 & & & & \\
\hline
x^8 & -\ 10\,x^7 & +\ 25\,x^6 & -\ 12\,x^5 & +\ 68\,x^4 & -\ 40\,x^3 & +\ 36\,x^2 & -\ 48\,x & +\ 16
\end{array}
$$

$\times)$

To conclude,

$$\left(x^4 - 5x^3 - 6\,x + 4\right)^2$$

$$= \; x^8 - 10x^7 + 25x^6 - 12x^5 + 68x^4 - 40x^3 + 36x^2 - 48x + 16.$$

Example 5. Let's expand $\left(x^4 + x^3 + x^2 + x + 1\right)^2$.

The same deal:

$$\begin{array}{rrrrrrrrrr}
 & & & & & x^4 & + \; x^3 & + \; x^2 & + \; x & + \; 1 \\
 & & & & & x^4 & + \; x^3 & + \; x^2 & + \; x & + \; 1 \\
\times) & & & & & & & & & \\
\hline
 & & & & & x^4 & + \; x^3 & + \; x^2 & + \; x & + \; 1 \\
 & & & & x^5 & + \; x^4 & + \; x^3 & + \; x^2 & + \; x & \\
 & & & x^6 & + \; x^5 & + \; x^4 & + \; x^3 & + \; x^2 & & \\
 & & x^7 & + \; x^6 & + \; x^5 & + \; x^4 & + \; x^3 & & & \\
 & x^8 & + \; x^7 & + \; x^6 & + \; x^5 & + \; x^4 & & & & \\
\hline
 & x^8 & + \; 2x^7 & + \; 3x^6 & + \; 4x^5 & + \; 5x^4 & + \; 4x^3 & + \; 3x^2 & + \; 2x & + \; 1
\end{array}$$

To conclude,

$$\left(x^4 + x^3 + x^2 + x + 1\right)^2$$

$$= \; x^8 + 2x^7 + 3x^6 + 4x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1.$$

Exercise 1. Expand

(1) $\left(x^3 - 8x\right)^2$. (2) $\left(2x^2 - 3x + 5\right)^2$.

(3) $\left(x^3 + x^2 - 4\right)^2$. (4) $\left(x^2 + \frac{1}{2}x + \frac{1}{3}\right)^2$.

(5) $\left(1 + x + x^2 + x^3 + x^4 + x^5\right)^2$.

$\left[\underline{\textbf{Answers}}\right]$**:**

(1) $x^6 - 16x^4 + 64x^2$. (2) $4x^4 - 12x^3 + 29x^2 - 30x + 25$.

(3) $x^6 + 2x^5 + x^4 - 8x^3 - 8x^2 + 16$.

(4) $x^4 + x^3 + \frac{11}{12}x^2 + \frac{11}{3}x + \frac{11}{9}$.

(5) $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 5x^6 + 4x^7 + 3x^8 + 2x^9 + x^{10}$.

- **Product of three or more polynomials.**

Example 6. How about

$$\left(x + 1\right)\left(x^2 + x + 1\right)\left(x^4 + x^2 + 1\right)?$$

There are three polynomials involved. This one you have to do it step by step, namely, you first do the boxed part:

$$\boxed{\left(x + 1\right)\left(x^2 + x + 1\right)}\ \left(x^4 + x^2 + 1\right)$$

and then you multiply the outcome with the third factor $x^4 + x^2 + 1$. Let's do it:

Step 1. Do $\Big(x + 1\Big)\Big(x^2 + x + 1\Big)$:

$$\begin{array}{rccccccc}
 & & & x^2 & + & x & + & 1 \\
 & & & & & x & + & 1 \\
\times) & & & & & & & \\
\hline
 & & & x^2 & + & x & + & 1 \\
 & x^3 & + & x^2 & + & x & & \\
\hline
 & x^3 & + & 2\,x^2 & + & 2\,x & + & 1
\end{array}$$

In short,

$$\Big(x + 1\Big)\Big(x^2 + x + 1\Big) \;=\; x^3 + 2x^2 + 2x + 1.$$

Step 2. Do $\Big(x^3 + 2x^2 + 2x + 1\Big)\Big(x^4 + x^2 + 1\Big)$:

$$\begin{array}{rccccccccccccccc}
 & & & & & & & x^4 & & & + & x^2 & & & + & 1 \\
 & & & & & & & & & x^3 & + & 2\,x^2 & + & 2\,x & + & 1 \\
\times) & & & & & & & & & & & & & & & \\
\hline
 & & & & & & & x^4 & & & + & x^2 & & & + & 1 \\
 & & & & & 2\,x^5 & & & + & 2\,x^3 & & & + & 2\,x & & \\
 & & & 2\,x^6 & & & + & 2\,x^4 & & & + & 2\,x^2 & & & & \\
 & x^7 & & & + & x^5 & & & + & x^3 & & & & & & \\
\hline
 & x^7 & + & 2\,x^6 & + & 3\,x^5 & + & 3\,x^4 & + & 3\,x^3 & + & 3\,x^2 & + & 2\,x & + & 1
\end{array}$$

In short,

$$\begin{aligned}
&\Big(x^3 + 2x^2 + 2x + 1\Big)\Big(x^4 + x^2 + 1\Big) \\
&\qquad = \; x^7 + 2x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 2x + 1.
\end{aligned}$$

To conclude,

$$\begin{aligned}\Big(x + 1\Big)\Big(x^2 + x + 1\Big)\Big(x^4 + x^2 + 1\Big)& \\ &= x^7 + 2x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 2x + 1.\end{aligned}$$

Exercise 2. Expand:

(1) $\Big(x - 1\Big)\Big(x + 1\Big)^2$.

(2) $\Big(x - 1\Big)\Big(x - 3\Big)\Big(x^2 - 3\Big)$.

(3) $\Big(x - 1\Big)\Big(x + 1\Big)\Big(x^2 + 1\Big)$.

(4) $\Big(x^2 - \sqrt{2}\,x + 1\Big)\Big(x^2 + \sqrt{2}\,x + 1\Big)\Big(x^2 - 1\Big)$.

(5) $\Big(x - \sqrt{2} - 1\Big)\Big(x + \sqrt{2} - 1\Big)\Big(x - \sqrt{2} + 1\Big)\Big(x + \sqrt{2} + 1\Big)$.

[**Answers**]**:**

(1) $x^3 + x^2 - x - 1$.

(2) $x^4 - 4x^3 + 12x - 9$.

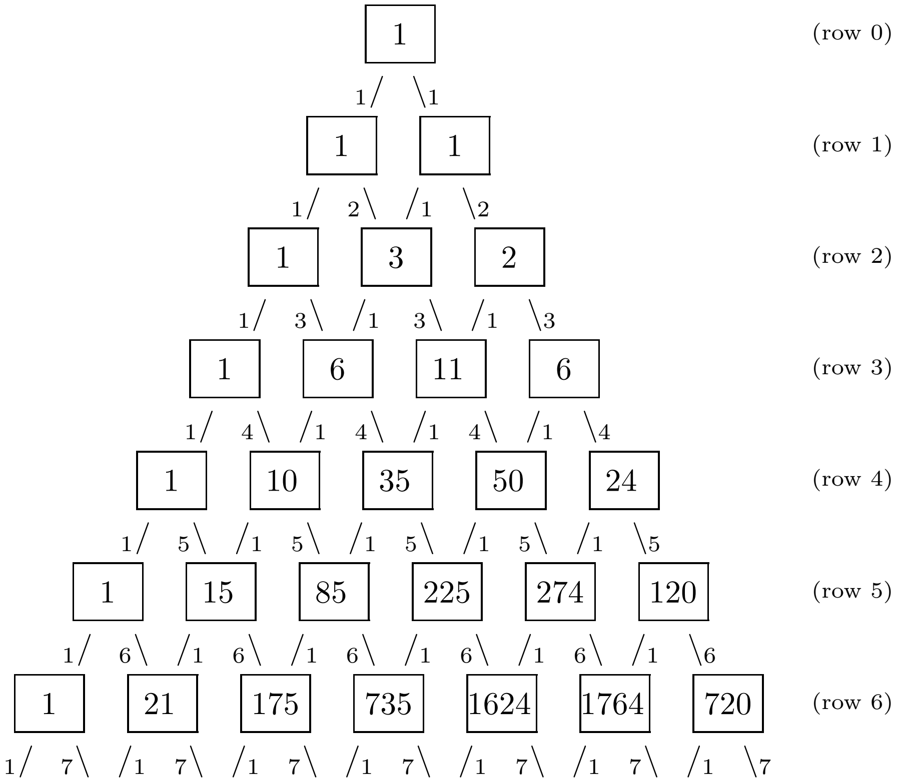
(3) $x^4 - 1$.

(4) $x^6 - x^4 + x^2 - 1$.

(5) $x^4 - 6\,x^2 + 1$.

- **Raising products.**

  The following has some bearings on certain types of polynomial multiplications:



The above — apparently a variation of Pascal — can be used to get the expansions

$$
\begin{aligned}
&\Big(x+1\Big),\\
&\Big(x+1\Big)\Big(x+2\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big)\Big(x+5\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big)\Big(x+5\Big)\Big(x+6\Big),\\
&\qquad\vdots \qquad\qquad\qquad\qquad\qquad\qquad \ddots
\end{aligned}
$$

Namely:

$$\Big(x + 1\Big)\Big(x + 2\Big) \;=\; x^2 + 3\,x + 2,$$

$$\Big(x + 1\Big)\Big(x + 2\Big)\Big(x + 3\Big) \;=\; x^3 + 6\,x^2 + 11\,x + 6,$$

$$\Big(x + 1\Big)\Big(x + 2\Big)\Big(x + 3\Big)\Big(x + 4\Big)$$

$$= x^4 + 10\,x^3 + 35\,x^2 + 50\,x + 24,$$

$$\Big(x + 1\Big)\Big(x + 2\Big)\Big(x + 3\Big)\Big(x + 4\Big)\Big(x + 5\Big)$$

$$= x^5 + 15\,x^4 + 85\,x^3 + 225\,x^2 + 274\,x + 120,$$

$$\Big(x + 1\Big)\Big(x + 2\Big)\Big(x + 3\Big)\Big(x + 4\Big)\Big(x + 5\Big)\Big(x + 6\Big)$$

$$= x^6 + 21\,x^5 + 175\,x^4 + 735\,x^3 + 1624\,x^2 + 1764\,x + 720.$$

Exercise 3. Expand:

$$\Big(x + 1\Big)\Big(x + 2\Big)\Big(x + 3\Big)\Big(x + 4\Big)\Big(x + 5\Big)\Big(x + 6\Big)\Big(x + 7\Big).$$

⋆ As for this, the triangle in the previous page is apparently shown only up to the sixth row. You have to extend it to the seventh row.

[**Answer**]**:**

$$x^7 + 28x^6 + 322x^5 + 1960x^4 + 6769x^3 + 13132x^2 + 13068x + 5040.$$