Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – XXIII

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§23. Polynomials and their arithmetic – II.

• **A monomial multipled to a polynomial.**

More generally, we may multiply a monomial with a polynomial. The outcome is a polynomial. This process is called an <u>expansion</u>.

Example 1. Let's expand

$$x\left(x^4 + 3x^3 + 5x^2\right).$$

This is simply raise the exponent in each x-to-the-power inside the parenthesis by 1. So

$$x\left(x^4 + 3x^3 + 5x^2\right) = x^5 + 3x^4 + 5x^3.$$

Example 2. Let's expand

$$x^2\left(8x^3 + 24x^2 + 6x + 3\right).$$

This time you raise the exponent in each x-to-the-power inside the parenthesis by 2. So

$$x^2\left(8x^3 + 24x^2 + 6x + 3\right) = 8x^5 + 24x^4 + 6x^3 + 3x^2.$$

Example 3. Let's expand

$$2x\left(x^2 + 8x + 5\right).$$

This requires you to do two things at once, namely, (i) raise the exponent in each x-to-the-power inside the parenthesis by 1, and then (ii) multiply 2 to each of the terms. So

$$2x\left(x^2 + 8x + 5\right) = 2x^3 + 16x^2 + 10x.$$

Example 4. Let's expand

$$5x^4\left(6x^5 + 12x^3 + 8x^2 + 7\right).$$

This is similar, do two things at once: namely, (i) raise the exponent in each x-to-the-power inside the parenthesis by 4, and then (ii) multiply 5 to each of the terms. So

$$5x^4\left(6x^5 + 12x^3 + 8x^2 + 7\right) = 30x^9 + 60x^7 + 40x^6 + 35x^4.$$

Exercise 1. Expand

(1) $x\left(x^2 - x + 4\right)$. (2) $x^3\left(2x^4 + 8x^3 + 5x\right)$.

(3) $6x\left(-\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{6}x\right)$.

(4) $\frac{1}{4}x^2\left(x^4 + 2x^3 + 3x^2 + 2x + 1\right)$.

[**Answers**]:

(1) $x^3 - x^2 + 4x$. (2) $2x^7 + 8x^6 + 5x^4$.

(3) $-2x^4 + 3x^3 - x^2$. (4) $\frac{1}{4}x^6 + \frac{1}{2}x^5 + \frac{3}{4}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2$.

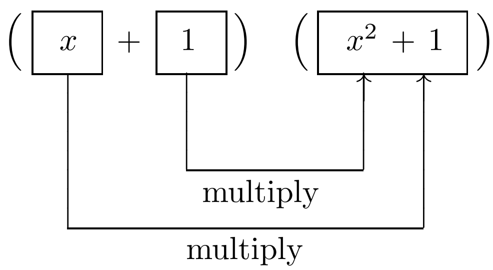
- **Multiplying out two polynomials.**

Even more generally, we may multiply out two polymials. The outcome is a polynomial. This process is once again called an expansion.

Example 5. Let's expand

$$\Big(x + 1\Big)\Big(x^2 + 1\Big).$$

It goes as follows.



$$= x\Big(x^2 + 1\Big) + 1\Big(x^2 + 1\Big)$$

$$= \Big(x^3 + x\Big) + \Big(x^2 + 1\Big)$$

$$= x^3 + x + x^2 + 1$$

$$= x^3 + x^2 + x + 1.$$

In short,

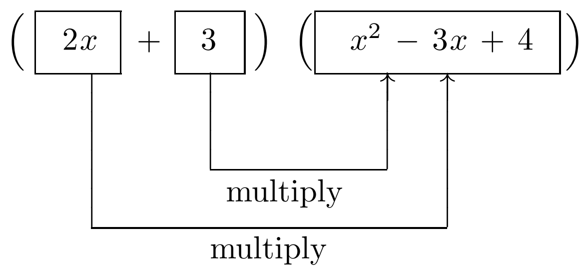
$$\Big(x + 1\Big)\Big(x^2 + 1\Big) = x^3 + x^2 + x + 1.$$

⋆ As you can see, the above consists of two multiplications, both "monomial times polynomial" type, followed by one polynomial addition.

Example 6. Let's expand

$$\Big(2x^2 + 3\Big)\Big(x^2 - 3x + 4\Big).$$

It goes as follows.



$$= \; 2x\,\Big(x^2 - 3x + 4\Big) \quad + \quad 3\,\Big(x^2 - 3x + 4\Big)$$

$$= \; \Big(2x^3 - 6x^2 + 8x\Big) \quad + \quad \Big(3x^2 - 9x + 12\Big)$$

$$= \; 2x^3 - 6x^2 + 8x + 3x^2 - 9x + 12$$

(uncovered parentheses)

$$= \; 2x^3 - 6x^2 + 3x^2 + 8x - 9x + 12$$

(re-ordered terms)

$$= \; 2x^3 - 3x^2 - x + 12.$$

In short,

$$\Big(2x + 3\Big)\Big(x^2 - 3x + 4\Big) \; = \; 2x^3 - 3x^2 - x + 12.$$

⋆ Once again, the procedure consists of two multiplications, both “monomial times polynomial” type, followed by one polynomial addition. The following recaptures the same algorithm:

$$
\begin{array}{lrrrrrrr}
 & & & x^2 & - & 3x & + & 4 \\
 & & & & & 2x & + & 3 \\
\times) & & & & & & & \\ \hline
 & & & 3x^2 & - & 9x & + & 12 \\
 & 2x^3 & - & 6x^2 & + & 8x & & \\ \hline
 & 2x^3 & - & 3x^2 & - & x & + & 12
\end{array}
$$

Let’s dissect. Take a look at the following which is a mere duplication of the above, but with some highlighted boxes:

$$
\begin{array}{lrrrrrrr}
 & & & \multicolumn{5}{r}{\boxed{x^2 \quad - \quad 3x \quad + \quad 4}} \\
 & & & & & 2x & \multicolumn{2}{r}{\boxed{+ \quad 3}} \\
\times) & & & & & & & \\ \hline
 & & & \multicolumn{5}{r}{\boxed{3x^2 \quad - \quad 9x \quad + \quad 12}} \\
 & 2x^3 & - & 6x^2 & + & 8x & & \\ \hline
 & 2x^3 & - & 3x^2 & - & x & + & 12
\end{array}
$$

Ignore the parts that are not enclosed by the boxes. Only look at those highlighted by the boxes. This is just constant 3 (which is a monomial) multiplied to the polynomial $x^2 - 3x + 4$. The outcome is highlighted in the lowest box. Next, look at the following, I have simply changed the locations of the boxes:

$$
\begin{array}{rrrrrrrr}
 & & \boxed{x^2} & \boxed{-} & \boxed{3x} & \boxed{+} & \boxed{4} \\
 & & & & \boxed{2x} & + & 3 \\
\times) & & & & & & \\
\hline
 & & 3x^2 & - & 9x & + & 12 \\
\boxed{2x^3} & \boxed{-} & \boxed{6x^2} & \boxed{+} & \boxed{8x} & & \\
\hline
2x^3 & - & 3x^2 & - & x & + & 12
\end{array}
$$

Again, ignore everything else but only look at the highlighted parts. This is just monomial $2x$ multiplied to the polynomial $x^2 - 3x + 4$. The outcome is highlighted in the lowest box. Finally, just one more time I'm going to change the locations of the boxes:

$$
\begin{array}{rrrrrrrr}
 & & x^2 & - & 3x & + & 4 \\
 & & & & 2x & + & 3 \\
\times) & & & & & & \\
\hline
\boxed{} & & \boxed{3x^2} & \boxed{-} & \boxed{9x} & \boxed{+} & \boxed{12} \\
\boxed{2x^3} & \boxed{-} & \boxed{6x^2} & \boxed{+} & \boxed{8x} & & \\
\hline
\boxed{2x^3} & \boxed{-} & \boxed{3x^2} & \boxed{-} & \boxed{x} & \boxed{+} & \boxed{12}
\end{array}
$$

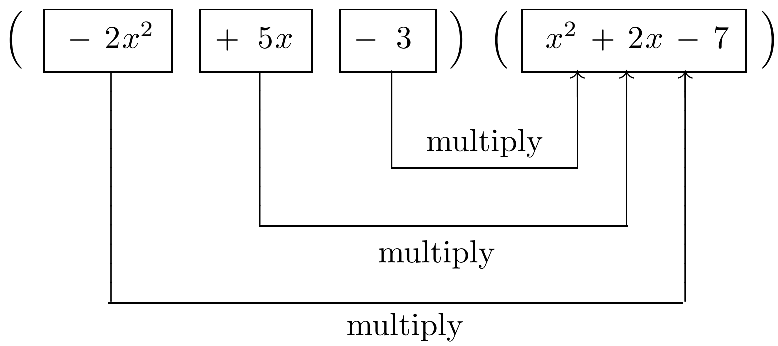
Only look at the last three lines as highlighted. This is just polynomial additions. The outcome is highlighted in the lowest line in the box. This is exactly the final outcome.

Let's do another example.

Example 7. Let's expand

$$\Big(-\,2x^2 \,+\, 5x \,-\, 3\Big)\Big(x^2 \,+\, 2x \,-\, 7\Big).$$

It goes as follows.



$$= \; -2x^2\,\Big(x^2 \,+\, 2x \,-\, 7\Big) \;+\; 5x\,\Big(x^2 \,+\, 2x \,-\, 7\Big) \;-\; 3\,\Big(x^2 \,+\, 2x \,-\, 7\Big)$$

$$= \; \Big(-\,2\,x^4 - 4x^3 + 14\,x^2\Big) \;+\; \Big(5\,x^3 + 10x^2 - 35\,x\Big) \;+\; \Big(-\,3x^2 - 6x + 21\Big)$$

$$= \; -2\,x^4 \,-\, 4x^3 \,+\, 14\,x^2 \,+\, 5\,x^3 \,+\, 10x^2 \,-\, 35\,x \,-\, 3x^2 \,-\, 6x \,+\, 21$$

$$\Big(\text{uncovered parentheses}\Big)$$

$$= \; -2x^4 \,-\, 4x^3 \,+\, 5x^3 \,+\, 14x^2 \,+\, 10x^2 \,-\, 3x^2 \,-\, 35x \,-\, 6x \,+\, 21$$

$$\Big(\text{re-ordered terms}\Big)$$

$$= \; -2x^4 \,+\, x^3 \,+\, 21x^2 \,-\, 41x \,+\, 21.$$

In short,

$$\Big(-\,2x^2 \,+\, 5x \,-\, 3\Big)\Big(x^2 \,+\, 2x \,-\, 7\Big) \;=\; -2x^4 \,+\, x^3 \,+\, 21x^2 \,-\, 41x \,+\, 21.$$

⋆ Just like the previous example, we can do it the following way:

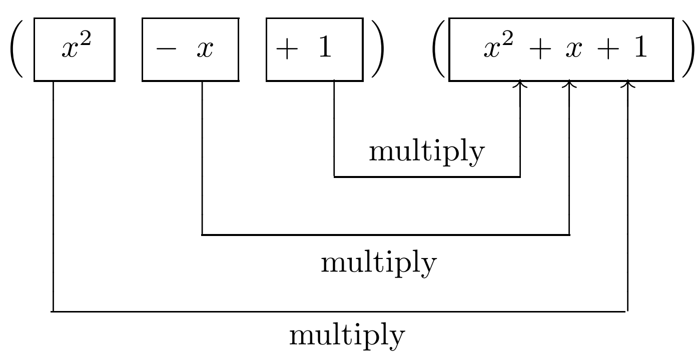
$$\begin{array}{rrrrrrrrr}
 & & & & x^2 & + & 2\,x & - & 7 \\
 & & & - & 2\,x^2 & + & 5\,x & - & 3 \\
\hline
 & & & - & 3\,x^2 & - & 6\,x & + & 21 \\
 & & 5\,x^3 & + & 10\,x^2 & - & 35\,x & & \\
-2\,x^4 & - & 4\,x^3 & + & 14\,x^2 & & & & \\
\hline
-2\,x^4 & + & x^3 & + & 21\,x^2 & - & 41\,x & + & 21
\end{array}
\quad (\times)$$

You can do it either way, whichever way suits you better. Let's do more examples.

Example 8. Let's expand

$$\left(x^2 - x + 1\right)\left(x^2 + x + 1\right).$$

It goes as follows.



$$
\begin{aligned}
&= x^2\Big(x^2+x+1\Big) - x\Big(x^2+x+1\Big) + 1\Big(x^2+x+1\Big) \\
&= \Big(x^4+x^3+x^2\Big) + \Big(-x^3-x^2-x\Big) + \Big(x^2+x+1\Big) \\
&= x^4+x^3+x^2-x^3-x^2-x+x^2+x+1 \\
&\qquad\qquad\qquad\qquad\Big(\text{uncovered parentheses}\Big) \\
&= x^4+x^3-x^3+x^2-x^2+x^2-x+x+1 \\
&\qquad\qquad\qquad\qquad\Big(\text{re-ordered terms}\Big) \\
&= x^4+x^2+1.
\end{aligned}
$$

In short,

$$\Big(x^2-x+1\Big)\Big(x^2+x+1\Big) = x^4+x^2+1.$$

⋆ The following is an alternative way:

$$
\begin{array}{rrrrrrrrr}
 & & & & x^2 & + & x & + & 1 \\
 & & & & x^2 & - & x & + & 1 \\
\hline
\times) & & & & x^2 & + & x & + & 1 \\
 & - & x^3 & - & x^2 & - & x & & \\
x^4 & + & x^3 & + & x^2 & & & & \\
\hline
x^4 & & & + & x^2 & & & + & 1
\end{array}
$$

- Let's do

$$(0) \qquad \Big(x - 1\Big) \cdot 1,$$

$$(1) \qquad \Big(x - 1\Big)\Big(x + 1\Big),$$

$$(2) \qquad \Big(x - 1\Big)\Big(x^2 + x + 1\Big),$$

$$(3) \qquad \Big(x - 1\Big)\Big(x^3 + x^2 + x + 1\Big),$$

$$(4) \qquad \Big(x - 1\Big)\Big(x^4 + x^3 + x^2 + x + 1\Big),$$

$$(5) \qquad \Big(x - 1\Big)\Big(x^5 + x^4 + x^3 + x^2 + x + 1\Big),$$

$$(6) \qquad \Big(x - 1\Big)\Big(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\Big),$$

$$\vdots$$

Let's try (5). Let's do it the second way:

$$\begin{array}{rccccccccccccc}
 & & & x^5 & + & x^4 & + & x^3 & + & x^2 & + & x & + & 1 \\
 & & & & & & & & & & & x & - & 1 \\
\times) & & & & & & & & & & & & & \\ \hline
 & & & -x^5 & - & x^4 & - & x^3 & - & x^2 & - & x & - & 1 \\
 & x^6 & + & x^5 & + & x^4 & + & x^3 & + & x^2 & + & x & & \\ \hline
 & x^6 & & & & & & & & & & & - & 1
\end{array}$$

In short,

$$(5) \qquad \Big(x - 1\Big)\Big(x^5 + x^4 + x^3 + x^2 + x + 1\Big) = x^6 - 1.$$

By extrapolation:

$$\begin{aligned}
&(0) \quad && \Big(x - 1\Big) \cdot 1 &&= x - 1,\\
&(1) \quad && \Big(x - 1\Big)\Big(x + 1\Big) &&= x^2 - 1,\\
&(2) \quad && \Big(x - 1\Big)\Big(x^2 + x + 1\Big) &&= x^3 - 1,\\
&(3) \quad && \Big(x - 1\Big)\Big(x^3 + x^2 + x + 1\Big) &&= x^4 - 1,\\
&(4) \quad && \Big(x - 1\Big)\Big(x^4 + x^3 + x^2 + x + 1\Big) &&= x^5 - 1,\\
&(5) \quad && \Big(x - 1\Big)\Big(x^5 + x^4 + x^3 + x^2 + x + 1\Big) &&= x^6 - 1,\\
&(6) \quad && \Big(x - 1\Big)\Big(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\Big) &&= x^7 - 1,
\end{aligned}$$

$$\vdots$$

$\star$ A minor tweak:

$$\begin{aligned}
&(0) \quad && \Big(1 - x\Big) \cdot 1 &&= 1 - x,\\
&(1) \quad && \Big(1 - x\Big)\Big(1 + x\Big) &&= 1 - x^2,\\
&(2) \quad && \Big(1 - x\Big)\Big(1 + x + x^2\Big) &&= 1 - x^3,\\
&(3) \quad && \Big(1 - x\Big)\Big(1 + x + x^2 + x^3\Big) &&= 1 - x^4,\\
&(4) \quad && \Big(1 - x\Big)\Big(1 + x + x^2 + x^3 + x^4\Big) &&= 1 - x^5,\\
&(5) \quad && \Big(1 - x\Big)\Big(1 + x + x^2 + x^3 + x^4 + x^5\Big) &&= 1 - x^6,\\
&(6) \quad && \Big(1 - x\Big)\Big(1 + x + x^2 + x^3 + x^4 + x^5 + x^6\Big) &&= 1 - x^7,
\end{aligned}$$

$$\vdots$$

$\star$ Further tweak:

$$
\begin{array}{lll}
(0) & \Big(1 + x\Big) \cdot 1 & = \ 1 + x, \\
(1) & \Big(1 + x\Big)\Big(1 - x\Big) & = \ 1 - x^2, \\
(2) & \Big(1 + x\Big)\Big(1 - x + x^2\Big) & = \ 1 + x^3, \\
(3) & \Big(1 + x\Big)\Big(1 - x + x^2 - x^3\Big) & = \ 1 - x^4, \\
(4) & \Big(1 + x\Big)\Big(1 - x + x^2 - x^3 + x^4\Big) & = \ 1 + x^5, \\
(5) & \Big(1 + x\Big)\Big(1 - x + x^2 - x^3 + x^4 - x^5\Big) & = \ 1 - x^6, \\
(6) & \Big(1 + x\Big)\Big(1 - x + x^2 - x^3 + x^4 - x^5 + x^6\Big) & = \ 1 + x^7, \\
 & \vdots &
\end{array}
$$

Exercise 2. Expand

(1) $\Big(x + 3\Big)\Big(x + 4\Big)$.

(2) $\Big(x + 5\Big)\Big(x - 2\Big)$.

(3) $\Big(x^2 - 7\Big)\Big(x^2 - 3x + 1\Big)$.

(4) $\Big(x^3 - 4x^2 + 2\Big)\Big(x^2 + 3x - 8\Big)$.

[**Answers**]:

(1) $x^2 + 7x + 12$.

(2) $x^2 + 3x - 10$.

(3) $x^4 - 3x^3 - 6x^2 + 21x - 7$.

(4) $x^5 - x^4 - 20x^3 + 34\,x^2 + 6x - 16$.

Exercise 3. Expand

(1) $$\left(x - 1\right)\Big(x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1\Big).$$

(2) $$\left(x - 1\right)\Big(x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1\Big).$$

(3) $$\left(1 - x\right)\Big(1 + x + x^{2} + x^{3} + x^{4} + x^{5} + x^{6} + x^{7} + x^{8} + x^{9} + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19} + x^{20} + x^{21} + x^{22} + x^{23}\Big).$$

(4) $$\left(1 + x\right)\Big(1 - x + x^{2} - x^{3} + x^{4} - x^{5} + x^{6} - x^{7} + x^{8} - x^{9} + x^{10} - x^{11} + x^{12} - x^{13} + x^{14}\Big).$$

(5) $$\left(1 + x\right)\Big(1 - x + x^{2} - x^{3} + x^{4} - x^{5} + x^{6} - x^{7} + x^{8} - x^{9} + x^{10} - x^{11} + x^{12} - x^{13} + x^{14} - x^{15} + x^{16} - x^{17} + x^{18} - x^{19} + x^{20} - x^{21} + x^{22} - x^{23} + x^{24} - x^{25} + x^{26} - x^{27} + x^{28} - x^{29} + x^{30} - x^{31} + x^{32} - x^{33} + x^{34} - x^{35} + x^{36} - x^{37} + x^{38} - x^{39} + x^{40} - x^{41} + x^{42} - x^{43} + x^{44} - x^{45} + x^{46} - x^{47}\Big).$$

[**Answers**]:

(1) $x^{21} - 1.$ (2) $x^{41} - 1.$

(3) $1 - x^{24}.$ (4) $1 + x^{15}.$

(5) $1 - x^{48}.$