Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XXII

March 25 (Wed), 2015

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§22. Polynomials and their arithmetic – I.

• Today I want to talk about polynomials. Polynomials are made of monomials. So I cover both. Spellings first:

"polynomial," "monomial."

First, we call each of

$$1, \quad x, \quad x^2, \quad x^3, \quad x^4, \quad x^5, \quad \cdots,$$

a monomial in x. Here, we include 1 in the list because $1 = x^0$. Next, we also call each of

$$3, \quad -2x, \quad 4x^2, \quad \sqrt{2}\,x^3, \quad \frac{1}{4}x^4, \quad 7x^5, \quad \cdots,$$

a monomial in x.

More generally, suppose a is a constant real number, and n is an integer with $n \geq 0$. Then we call

$$a\,x^n$$

a monomial in x.

Note that neither of

$$\boxed{x^{-1}} \qquad \text{nor} \qquad \boxed{x^{\frac{1}{2}}}$$

is a monomial, because the exponent is either negative or a non-integer.

Next, a <u>polynomial</u> is a finite sum of monomials. Thus

$$1 + x, \quad 2x + x^2, \quad -\frac{4}{5} + \sqrt{3}x^3, \quad x + x^4 + x^7, \quad -6 + 2x - 8x^3 + x^5,$$

are examples of polynomials in x. On the other hand, none of

$$\sqrt{x}, \qquad 1 + x^{\frac{3}{2}}, \qquad \frac{1}{x}, \qquad \sqrt{2 + x^2}, \qquad \frac{3}{4 - x}, \qquad 2x^{-2} + x^2$$

is a polynomial in x.

• **Note.** There is no general rule that we must obey when it comes to the order of terms in polynomial expressions. So, you may write either

$$2 + x, \qquad \text{or} \qquad x + 2.$$

These two are one and the same. Similarly, you may write either

$$4 - 3x + x^2, \qquad -3x + x^2 + 4, \qquad x^2 + 4 - 3x,$$

$$4 + x^2 - 3x, \qquad x^2 - 3x + 4, \qquad \text{or} \qquad -3x + 4 + x^2.$$

These six are all one and the same. Usually, though, we prefer to write polynomials either in the <u>ascending order</u> or in the <u>descending order</u> of exponents. So,

$$4 - 3x + x^2 \qquad \Big(\text{ascending order}\Big),$$

and

$$x^2 - 3x + 4 \qquad \Big(\text{descending order}\Big),$$

are equally preferable.

Exercise 1. Permute the order of terms, if necessary, to make each of the given polynomials in the ascending order.

(1) $2x + x^4 - \frac{1}{2}x^2.$ (2) $-\frac{4}{3}x^3 - 5x^2 - 4x^5.$

(3) $x^8 + 5x^6 - 10x^9 + 3.$ (4) $x + \sqrt{5}\,x^5 + \sqrt{3}\,x^3 - \sqrt{2}\,x^2.$

(5) $x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$

[**Answers**]:

(1) $2x - \frac{1}{2}x^2 + x^4.$ (2) $-5x^2 - \frac{4}{3}x^3 - 4x^5.$

(3) $3 + 5x^6 + x^8 - 10x^9.$ (4) $x - \sqrt{2}\,x^2 + \sqrt{3}\,x^3 + \sqrt{5}\,x^5.$

(5) $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9.$

Exercise 2. Permute the order of terms, if necessary, to make each of the given polynomials in the descending order.

(1) $6x^2 + x^3.$ (2) $\frac{1}{2}x^3 + \frac{1}{3}x^4 - x.$

(3) $x^7 + 5x^5 - 10x^8 + 4x^6.$ (4) $1 - x^2 + x^4 - x^6 + x^8 - x^{10}.$

[**Answers**]:

(1) $x^3 + 6x^2.$ (2) $\frac{1}{3}x^4 + \frac{1}{2}x^3 - x.$

(3) $-10x^8 + x^7 + 4x^6 + 5x^5.$ (4) $-x^{10} + x^8 - x^6 + x^4 - x^2 + 1.$

- **Polynomial addition.**

We may add two polynomials and the outcome is a polynomial. Let me use some examples to illustrate it.

Example 1. Let's calculate

$$\Big(x + 2\Big) + \Big(x^3 + 4x\Big).$$

This is just uncover the parentesis and reorder the terms $\big($in either the ascending or descending order$\big)$. So,

$$\begin{aligned}\Big(x^2 + 2\Big) + \Big(x^3 + 4x\Big) &= x^2 + 2 + x^3 + 4x \\ &= x^3 + x^2 + 4x + 2.\end{aligned}$$

$\big($This answer is clearly written in the descending order. You can write it in the ascending order instead. You don't have to give both.$\big)$

Example 2. Let's calculate

$$\left(x^4 + \frac{1}{5}x^2\right) + \Big(2x^4 - 6x^3 + x\Big).$$

This one looks pretty similar to Example 1 above, but this one involves more than what was required in Example 1. Indeed, let's just first uncover the parenthesis, and reorder terms:

$$\begin{aligned}&\left(x^4 + \frac{1}{5}x^2\right) + \Big(2x^4 - 6x^3 + x\Big) \\ &= x^4 + \frac{1}{5}x^2 + 2x^4 - 6x^3 + x \\ &= x^4 + 2x^4 - 6x^3 + \frac{1}{5}x^2 + x.\end{aligned}$$

Realize that there are two monomials that involve x^4. You have to combine them. x^4 and $2x^4$ make $3x^4$. So the above is simplified as

$$3x^4 - 6x^3 + \frac{1}{5}x^2 + x.$$

This is the final answer.

⋆ If you like, you can do the above as follows:

$$\begin{array}{rcccccccc} & x^4 & & & + & \frac{1}{5}x^2 & & & \\ & 2x^4 & - & 6x^3 & & & + & x & \\ +) \hline & 3x^4 & - & 6x^3 & + & \frac{1}{5}x^2 & + & x. & \end{array}$$

⋆ Let's do a similar example, but in a different format.

Example 3. Let's find $f(x) + g(x)$, where

$$f(x) = x^4 + \frac{7}{2}x^3 - \frac{5}{2}x^2 - x, \qquad \text{and}$$

$$g(x) = 6x^4 + \frac{1}{2}x^3 - x^2 - 3x + 2.$$

This is essentially the same type of a problem as the previous ones, but the difference is that two polynomials are given names as $f(x)$ and $g(x)$. Still, the same method works. Here is how it goes:

$f(x) + g(x)$

$$= \left(x^4 + \frac{7}{2}x^3 - \frac{5}{2}x^2 - x\right) + \left(6x^4 + \frac{1}{2}x^3 - x^2 - 3x + 2\right)$$

$$= x^4 + \frac{7}{2}x^3 - \frac{5}{2}x^2 - x + 6x^4 + \frac{1}{2}x^3 - x^2 - 3x + 2$$

(uncovered parentheses)

$$= x^4 + 6x^4 + \frac{7}{2}x^3 + \frac{1}{2}x^3 - \frac{5}{2}x^2 - x^2 - x - 3x + 2$$

(re-ordered terms)

$$= \left(x^4 + 6x^4\right) + \left(\frac{7}{2}x^3 + \frac{1}{2}x^3\right) + \left(-\frac{5}{2}x^2 - x^2\right) + \left(-x - 3x\right) + 2$$

$$= 7x^4 + 4x^3 + \left(-\frac{7}{2}x^2\right) + \left(-4x\right) + 2$$

$$= 7x^4 + 4x^3 - \frac{7}{2}x^2 - 4x + 2.$$

⋆ If you like, you can do the above as follows:

$$\begin{array}{rrrrrrrrrr} & x^4 & + & \frac{7}{2}x^3 & - & \frac{5}{2}x^2 & - & x & & \\ & 6x^4 & + & \frac{1}{2}x^3 & - & x^2 & - & 3x & + & 2 \\ +) & & & & & & & & & \\ \hline & 7x^4 & + & 4\ x^3 & - & \frac{7}{2}x^2 & - & 4x & + & 2 \end{array}$$

Exercise 3. Do

(1) $\left(x^7 + 3\,x^5 + 2\,x^3\right) + \left(-\,x^6 - x^4 - 2\,x^2\right).$

(2) $\left(x^4 + 9\,x^3 + 1\right) + \left(-\,x^4 - x^3 - 5\,x^2 + 2x + 3\right).$

(3) $\left(\frac{1}{2}x^3 + \frac{1}{3}x\right) + \left(\frac{1}{3}x^3 - \frac{1}{4}x\right).$

(4) $f\left(x\right) + g\left(x\right),$ where

$$f\left(x\right) = x^6 + 8x^5 + 12x^4 + 36x^3 + 9x^2,$$

$$g\left(x\right) = x^8 - 3x^6 - 8x^4 - 24x^3 + 45x - 120.$$

(5) $f\left(x\right) + g\left(x\right),$ where

$$f\left(x\right) = x^7 + x^5 + x^3 + x,$$

$$g\left(x\right) = x^8 + x^6 + x^4 + x^2 + 1.$$

[**Answers**]:

(1) $x^7 - x^6 + 3x^5 - x^4 + 2x^3 - 2\,x^2.$

(2) $8\,x^3 - 5x^2 + 2x + 4.$

(3) $\frac{5}{6}\,x^3 + \frac{1}{12}\,x.$

(4) $x^8 - 2x^6 + 8x^5 + 4x^4 + 12x^3 + 9x^2 + 45x - 120.$

(5) $x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$

- **Polynomial subtraction.**

We may subtract one polynomial from another polynomial. The outcome is a polynomial. This is very similar to polynomial additions. Let's do some example.

Example 4. Let's do

$$\Big(x^3 + 2x^2\Big) - \Big(3x + 4\Big).$$

When you uncover the parenteses, you have to be careful. Namely:

$$\Big(x^3 + 2x^2\Big) - \Big(3x + 4\Big) = x^3 + 2x^2 - 3x - 4.$$

In the above, notice that all the terms inside the second parenthesis got negated after uncovering that parenthesis. That's the correct way to do it.

Example 5. Let's do

$$\Big(x^6 - 8x^4 + 3\,x^2\Big) - \Big(2x^6 - 3x^4\Big).$$

Once again, when you uncover the parenteses,

$$\begin{aligned}
&\Big(x^6 - 8x^4 + 3\,x^2\Big) - \Big(2x^6 - 3x^4\Big) \\
&\qquad = x^6 - 8x^4 + 3x^2 - 2x^6 + 3x^4 \\
&\qquad \Big(\text{all the terms in the second parenthesis got negated}\Big) \\
&\qquad = x^6 - 2x^6 - 8x^4 + 3x^4 + 3x^2 \\
&\qquad \qquad \Big(\text{re-ordered terms}\Big) \\
&\qquad = \Big(x^6 - 2x^6\Big) + \Big(-8x^4 + 3x^4\Big) + 3x^2 \\
&\qquad = -x^6 - 5x^4 + 3x^2.
\end{aligned}$$

⋆ If you like, you can do it like

$$
\begin{array}{rrrrrr}
 & x^6 & - & 8x^4 & + & 3x^2 \\
 & 2x^6 & - & 3x^4 & & \\
-) & \hline -x^6 & - & 5x^4 & + & 3x^2
\end{array}
$$

Exercise 4. Do

(1) $\left(x^3 + 11\,x^2 + 21\,x\right) - \left(-\,x^2 - x + 4\right)$.

(2) $\left(-\,x^7 + 5x^6 + x^3 - 6\right) - \left(-\,2x^6 - 3x^4 + 7\,x^3 + 2x + 5\right)$.

(3) $\left(\dfrac{3}{2}x^4 + \dfrac{7}{4}x^2\right) - \left(\dfrac{1}{6}x^4 - \dfrac{1}{4}x^2 + 1\right)$.

(4) $f\left(x\right) - g\left(x\right)$, where

$$
\begin{aligned}
f\left(x\right) &= x^5 + 4x^4 + 16x^3 + 22x^2 + 18x, \\
g\left(x\right) &= x^6 - 31x^4 - 62x^2 + 72x + 56.
\end{aligned}
$$

(5) $f\left(x\right) - g\left(x\right)$, where

$$
f\left(x\right) = x^{13} + x^9 + x^5 + x, \qquad g\left(x\right) = x^{11} + x^7 + x^3.
$$

[**Answers**]:

(1) $x^3 + 12x^2 + 22x - 4$. (2) $-\,x^7 + 7x^6 + 3x^4 - 6x^3 - 2x - 11$.

(3) $\dfrac{4}{3}x^4 + 2x^2 - 1$. (4) $-x^6 + x^5 + 35x^4 + 16x^3 + 84x^2 - 54x - 56$.

(5) $x^{13} - x^{11} + x^9 - x^7 + x^5 - x^3 + x$.

- **Constsant multiplication.**

We may multiply a constant with a polynomial.

Example 6. Let's do

$$10\left(2\,x^3 + 3x^2 + 4\,x + 5\right).$$

This is simply multiply 10 to each of the terms. So

$$10\left(2\,x^3 + 3x^2 + 4\,x + 5\right) = 20\,x^3 + 30x^2 + 40\,x + 50.$$

Example 7. Let's do

$$-2\left(12\,x^5 - 21x^3 + 48\,x\right).$$

This is simply multiply -2 to each of the terms. So

$$-2\left(12\,x^5 - 21x^3 + 48\,x\right) = -24\,x^5 + 42x^3 - 96\,x.$$

Example 8. Sometimes you see something like

$$\frac{2x^4 - 11x^2 + 14x - 9}{2}.$$

This is the same as

$$\frac{1}{2}\left(2x^4 - 11x^2 + 14x - 9\right).$$

The answer is, of course,

$$x^4 - \frac{11}{2}x^2 + 7x - \frac{9}{2}.$$

Exercise 5. Simplify:

(1) $6\left(x^7 + 7\,x^6 + 21\,x^5\right)$.

(2) $-4\left(-\,x^2 + 5x + 3\right)$.

(3) $\dfrac{8x^{10} - 20x^8 + 24x^6 - 12x^4}{4}$.

(4) $\dfrac{1}{3}\left(\dfrac{3}{5}x^4 + \dfrac{3}{7}x^3 + \dfrac{3}{25}x^2 + \dfrac{3}{65}x\right)$.

(5) $3\,f\left(x\right)$ where

$$f\left(x\right) = x^5 + 4x^4 + 16x^3 + 22x^2 + 18x.$$

[**Answers**]:

(1) $6x^7 + 42x^6 + 126x^5$.

(2) $4x^2 - 20x - 12$.

(3) $2x^{10} - 5x^8 + 6x^6 - 3x^4$.

(4) $\dfrac{1}{5}x^4 + \dfrac{1}{7}x^3 + \dfrac{1}{25}x^2 + \dfrac{1}{65}x$.

(5) $3x^5 + 12x^4 + 48x^3 + 66x^2 + 54x$.