Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XXI

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Line #: 52920.

§21. Sum of consecutive cube numbers.

- Today I want to do

(1) $1^3 = ?$

(2) $1^3 + 2^3 = ?$

(3) $1^3 + 2^3 + 3^3 = ?$

(4) $1^3 + 2^3 + 3^3 + 4^3 = ?$

(5) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = ?$

(6) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = ?$

(7) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = ?$

(8) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 = ?$

(9) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 = ?$

This is a cube version of what we dealt with last time. So basically the same startegy should work. We want to concoct a formula for this so we can handle something like

(100)
$$\begin{aligned} & 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 \\ & + 11^3 + 12^3 + 13^3 + 14^3 + 15^3 + 16^3 + 17^3 + 18^3 + 19^3 + 20^3 \\ & + 21^3 + 22^3 + 23^3 + 24^3 + 25^3 + 26^3 + 27^3 + 28^3 + 29^3 + 30^3 \\ & + 31^3 + 32^3 + 33^3 + 34^3 + 35^3 + 36^3 + 37^3 + 38^3 + 39^3 + 40^3 \\ & + 41^3 + 42^3 + 43^3 + 44^3 + 45^3 + 46^3 + 47^3 + 48^3 + 49^3 + 50^3 \\ & + 51^3 + 52^3 + 53^3 + 54^3 + 55^3 + 56^3 + 57^3 + 58^3 + 59^3 + 60^3 \\ & + 61^3 + 62^3 + 63^3 + 64^3 + 65^3 + 66^3 + 67^3 + 68^3 + 69^3 + 70^3 \\ & + 71^3 + 72^3 + 73^3 + 74^3 + 75^3 + 76^3 + 77^3 + 78^3 + 79^3 + 80^3 \\ & + 81^3 + 82^3 + 83^3 + 84^3 + 85^3 + 86^3 + 87^3 + 88^3 + 89^3 + 90^3 \\ & + 91^3 + 92^3 + 93^3 + 94^3 + 95^3 + 96^3 + 97^3 + 98^3 + 99^3 + 100^3 = ? \end{aligned}$$

But how?

One way is to rely on the following:

Clue.

$$6\binom{n+2}{3} - 6\binom{n+1}{2} + \binom{n}{1} = n^3 .$$

What does this mean? Where does this come from? How is this useful? Don't worry, I will tell you everything. I will show you how to calculate (6) in a non brute-force way, using this information. It goes as follows. Let's start with the following which is a consequence of Pascal:

$$\begin{aligned}
&\binom{3}{3} = 1 = \binom{3}{3},\\
&\binom{3}{3} + \binom{4}{3} = 5 = \binom{5}{4},\\
&\binom{3}{3} + \binom{4}{3} + \binom{5}{3} = 15 = \binom{6}{4},\\
&\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} = 35 = \binom{7}{4},\\
&\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} + \binom{7}{3} = 70 = \binom{8}{4},\\
&\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} + \binom{7}{3} + \binom{8}{3} = 126 = \binom{9}{4}.
\end{aligned}$$

To spell each line out:

$$\begin{aligned}
&\frac{1\cdot2\cdot3}{6} = 1,\\
&\frac{1\cdot2\cdot3}{6} + \frac{2\cdot3\cdot4}{6} = 5,\\
&\frac{1\cdot2\cdot3}{6} + \frac{2\cdot3\cdot4}{6} + \frac{3\cdot4\cdot5}{6} = 15,\\
&\frac{1\cdot2\cdot3}{6} + \frac{2\cdot3\cdot4}{6} + \frac{3\cdot4\cdot5}{6} + \frac{4\cdot5\cdot6}{6} = 35,\\
&\frac{1\cdot2\cdot3}{6} + \frac{2\cdot3\cdot4}{6} + \frac{3\cdot4\cdot5}{6} + \frac{4\cdot5\cdot6}{6} + \frac{5\cdot6\cdot7}{6} = 70,\\
&\frac{1\cdot2\cdot3}{6} + \frac{2\cdot3\cdot4}{6} + \frac{3\cdot4\cdot5}{6} + \frac{4\cdot5\cdot6}{6} + \frac{5\cdot6\cdot7}{6} + \frac{6\cdot7\cdot8}{6} = 126.
\end{aligned}$$

Multiply 6 to the two sides in each line, to get rid of the denominators:

$$1\cdot 2\cdot 3 \;=\; 6 = 6\binom{4}{4},$$

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 \;=\; 30 = 6\binom{5}{4},$$

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 \;=\; 90 = 6\binom{6}{4},$$

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + 4\cdot 5\cdot 6 \;=\; 210 = 6\binom{7}{4},$$

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + 4\cdot 5\cdot 6 + 5\cdot 6\cdot 7 \;=\; 420 = 6\binom{8}{4},$$

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + 4\cdot 5\cdot 6 + 5\cdot 6\cdot 7 + 6\cdot 7\cdot 8 \;=\; 756 = 6\binom{9}{4}.$$

So the sixth line is

$$(A)_6 \qquad 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + 4\cdot 5\cdot 6 + 5\cdot 6\cdot 7 + 6\cdot 7\cdot 8 \;=\; 756.$$

Let's write it this way:

$$\begin{array}{ccccccccccccccc}
 & \underbrace{1\cdot 2\cdot 3} & + & \underbrace{2\cdot 3\cdot 4} & + & \underbrace{3\cdot 4\cdot 5} & + & \underbrace{4\cdot 5\cdot 6} & + & \underbrace{5\cdot 6\cdot 7} & + & \underbrace{6\cdot 7\cdot 8} & & \\
 & \| & & \| & & \| & & \| & & \| & & \| & & \\
(A)_6 & \boxed{6} & + & \boxed{24} & + & \boxed{60} & + & \boxed{120} & + & \boxed{210} & + & \boxed{336} & = & \boxed{756}
\end{array}$$

Meanwhile,

$$(B)_6 \qquad 3\cdot 1\cdot 2 + 3\cdot 2\cdot 3 + 3\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + 3\cdot 5\cdot 6 + 3\cdot 6\cdot 7 \;=\; 336.$$

$$\Big(\text{This is } 6 \text{ times} \quad \frac{1\cdot 2}{2} + \frac{2\cdot 3}{2} + \frac{3\cdot 4}{2} + \frac{4\cdot 5}{2} + \frac{5\cdot 6}{2} + \frac{6\cdot 7}{2} = 56.\Big)$$

Let's write it this way:

$$\underbrace{3\cdot 1\cdot 2}_{\|} + \underbrace{3\cdot 2\cdot 3}_{\|} + \underbrace{3\cdot 3\cdot 4}_{\|} + \underbrace{3\cdot 4\cdot 5}_{\|} + \underbrace{3\cdot 5\cdot 6}_{\|} + \underbrace{3\cdot 6\cdot 7}_{\|}$$

$$(\mathrm{B})_6 \quad \boxed{6} + \boxed{18} + \boxed{36} + \boxed{60} + \boxed{90} + \boxed{126} = \boxed{336}$$

Subtraction side by side $(\mathrm{A})_6 - (\mathrm{B})_6$:

$$\begin{array}{lccccccccccccc}
(\mathrm{A})_6 & \boxed{6} & + & \boxed{24} & + & \boxed{60} & + & \boxed{120} & + & \boxed{210} & + & \boxed{336} & = & \boxed{756} \\
(\mathrm{B})_6 & \boxed{6} & + & \boxed{18} & + & \boxed{36} & + & \boxed{60} & + & \boxed{90} & + & \boxed{126} & = & \boxed{336} \\
-) & & & & & & & & & & & & & \\ \hline
(\mathrm{C})_6 & \boxed{0} & + & \boxed{6} & + & \boxed{24} & + & \boxed{60} & + & \boxed{120} & + & \boxed{210} & = & \boxed{420}
\end{array}$$

Meanwhile

$$(\mathrm{D})_6 \quad 1 + 2 + 3 + 4 + 5 + 6 = 21.$$

Addition side by side $(\mathrm{C})_6 + (\mathrm{D})_6$:

$$\begin{array}{lccccccccccccc}
(\mathrm{C})_6 & \boxed{0} & + & \boxed{6} & + & \boxed{24} & + & \boxed{60} & + & \boxed{120} & + & \boxed{210} & = & \boxed{420} \\
(\mathrm{D})_6 & \boxed{1} & + & \boxed{2} & + & \boxed{3} & + & \boxed{4} & + & \boxed{5} & + & \boxed{6} & = & \boxed{21} \\
+) & & & & & & & & & & & & & \\ \hline
 & \boxed{1} & + & \boxed{8} & + & \boxed{27} & + & \boxed{64} & + & \boxed{125} & + & \boxed{216} & = & \boxed{441}
\end{array}$$

Just look at the last line. Let's duplicate:

$$\boxed{1} + \boxed{8} + \boxed{27} + \boxed{64} + \boxed{125} + \boxed{216} = \boxed{441}$$

Realize that each number inside the box on the left-hand side is a cube number. So, the answer is found. Namely:

$$1^3 \ + \ 2^3 \ + \ 3^3 \ + \ 4^3 \ + \ 5^3 \ + \ 6^3 \quad = \quad 441.$$

Now, that was for line (6) in the list

(1) $1^3 = ?$

(2) $1^3 + 2^3 = ?$

(3) $1^3 + 2^3 + 3^3 = ?$

(4) $1^3 + 2^3 + 3^3 + 4^3 = ?$

(5) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = ?$

(6) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = ?$

(7) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = ?$

(8) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 = ?$

(9) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 = ?$

But how about line (100), for example? We need to establish a formula. Basically, it suffices to find each of $(\mathrm{A})_n$, $(\mathrm{B})_n$, and $(\mathrm{D})_n$, the equivalent counterparts of $(\mathrm{A})_6$, $(\mathrm{B})_6$, and $(\mathrm{D})_6$ above. You call $(\mathrm{A})_n - (\mathrm{B})_n$ as $(\mathrm{C})_n$, and the answer will be $(\mathrm{C})_n + (\mathrm{D})_n$. So, in other words, the answer will just be

$$(\mathrm{A})_n \ - \ (\mathrm{B})_n \ + \ (\mathrm{D})_n.$$

Now, this is actually easy. Indeed,

$$(\mathrm{A})_n \qquad 6\binom{3}{3} + 6\binom{4}{3} + 6\binom{5}{3} + 6\binom{6}{3} + \cdots + 6\binom{n+2}{3} = 6\binom{n+3}{4}.$$

$$(\mathrm{B})_n \qquad 6\binom{2}{2} + 6\binom{3}{2} + 6\binom{4}{2} + 6\binom{5}{2} + \cdots + 6\binom{n+1}{2} = 6\binom{n+2}{3}.$$

$$(\mathrm{D})_n \qquad \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} + \cdots + \binom{n}{n} = \binom{n+1}{2}.$$

So, perform $(\mathrm{A})_n - (\mathrm{B})_n + (\mathrm{D})_n$ side by side. The left-hand side of the outcome is the sum of the following quantities:

$$6\binom{3}{3} - 6\binom{2}{2} + \binom{1}{1},$$

$$6\binom{4}{3} - 6\binom{3}{2} + \binom{2}{1},$$

$$6\binom{5}{3} - 6\binom{4}{2} + \binom{3}{1},$$

$$6\binom{6}{3} - 6\binom{5}{2} + \binom{4}{1},$$

$$\vdots$$

$$6\binom{n+2}{3} - 6\binom{n+1}{2} + \binom{n}{1}.$$

These are exactly

$$6\binom{3}{3} - 6\binom{2}{2} + \binom{1}{1} = 1^3,$$

$$6\binom{4}{3} - 6\binom{3}{2} + \binom{2}{1} = 2^3,$$

$$6\binom{5}{3} - 6\binom{4}{2} + \binom{3}{1} = 3^3,$$

$$6\binom{6}{3} - 6\binom{5}{2} + \binom{4}{1} = 4^3,$$

$$\vdots$$

$$6\binom{n+2}{3} - 6\binom{n+1}{2} + \binom{n}{1} = n^3.$$

And that is exactly the content of what I previously referred to as 'Clue' (in page 2). Meanwhile, we are yet to simplify the right-hand side of $(\mathrm{A})_n - (\mathrm{B})_n + (\mathrm{D})_n$. Let's duplicate $(\mathrm{A})_n$, $(\mathrm{B})_n$ and $(\mathrm{D})_n$:

$$(\mathrm{A})_n \qquad 6\binom{3}{3} + 6\binom{4}{3} + 6\binom{5}{3} + 6\binom{6}{3} + \cdots + 6\binom{n+2}{3} = 6\binom{n+3}{4}.$$

$$(\mathrm{B})_n \qquad 6\binom{2}{2} + 6\binom{3}{2} + 6\binom{4}{2} + 6\binom{5}{2} + \cdots + 6\binom{n+1}{2} = 6\binom{n+2}{3}.$$

$$(\mathrm{D})_n \qquad \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} + \cdots + \binom{n}{n} = \binom{n+1}{2}.$$

So the right-hand side of $(\mathrm{A})_n - (\mathrm{B})_n + (\mathrm{D})_n$ is

$$6\binom{n+3}{4} - 6\binom{n+2}{3} + \binom{n+1}{2}$$

$$= 6 \cdot \frac{n(n+1)(n+2)(n+3)}{4 \cdot 3 \cdot 2} - 6 \cdot \frac{n(n+1)(n+2)}{3 \cdot 2} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(n+2)(n+3)}{4} - n(n+1)(n+2) + \frac{n(n+1)}{2}$$

$$= n(n+1) \cdot \left(\frac{(n+2)(n+3)}{4} - (n+2) + \frac{1}{2} \right)$$

$$= n(n+1) \cdot \left(\frac{n^2+5n+6}{4} - \frac{4n+8}{4} + \frac{2}{4} \right)$$

$$= n(n+1) \cdot \frac{n^2+5n+6-4n-8+2}{4}$$

$$= n(n+1) \cdot \frac{n^2+n}{4}$$

$$= \frac{n(n+1)n(n+1)}{4} = \frac{n^2(n+1)^2}{4}.$$

So we came up with the following formula:

Formula. Let n be a positive integer. Then the sum of the cubes of the first n consecutive integers is

$$\boxed{1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 \quad = \quad \frac{1}{4}\, n^2 \Big(n+1\Big)^2.}$$

Now we can do line (100) easily using this formula:

$$\begin{aligned}
(100) \quad & 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 \\
& + 11^3 + 12^3 + 13^3 + 14^3 + 15^3 + 16^3 + 17^3 + 18^3 + 19^3 + 20^3 \\
& + 21^3 + 22^3 + 23^3 + 24^3 + 25^3 + 26^3 + 27^3 + 28^3 + 29^3 + 30^3 \\
& + 31^3 + 32^3 + 33^3 + 34^3 + 35^3 + 36^3 + 37^3 + 38^3 + 39^3 + 40^3 \\
& + 41^3 + 42^3 + 43^3 + 44^3 + 45^3 + 46^3 + 47^3 + 48^3 + 49^3 + 50^3 \\
& + 51^3 + 52^3 + 53^3 + 54^3 + 55^3 + 56^3 + 57^3 + 58^3 + 59^3 + 60^3 \\
& + 61^3 + 62^3 + 63^3 + 64^3 + 65^3 + 66^3 + 67^3 + 68^3 + 69^3 + 70^3 \\
& + 71^3 + 72^3 + 73^3 + 74^3 + 75^3 + 76^3 + 77^3 + 78^3 + 79^3 + 80^3 \\
& + 81^3 + 82^3 + 83^3 + 84^3 + 85^3 + 86^3 + 87^3 + 88^3 + 89^3 + 90^3 \\
& + 91^3 + 92^3 + 93^3 + 94^3 + 95^3 + 96^3 + 97^3 + 98^3 + 99^3 + 100^3
\end{aligned}$$

$$= \frac{100^2 \cdot \Big(100+1\Big)^2}{4}$$

$$= \frac{10000 \cdot 10201}{4}$$

$$= \frac{102010000}{4} = 25502500.$$

Exercise 1. Use formula above to find

$$(25) \qquad 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 + 11^3 + 12^3 + 13^3 + 14^3 + 15^3 + 16^3 + 17^3 + 18^3 + 19^3 + 20^3 + 21^3 + 22^3 + 23^3 + 24^3 + 25^3.$$

Solution By formula, this equals

$$\frac{25^2 \cdot \left(25+1\right)^2}{4} = \frac{625 \cdot 676}{4} = 625 \cdot 169 = 105625.$$

Exercise 2. How much does it make if you add up the cubes of integers between 1 and 500?

Solution By formula, this equals

$$\frac{500^2 \cdot \left(500+1\right)^2}{4} = \frac{250000 \cdot 251001}{4} = 15687562500.$$

• Now, in the above process of pulling Formula, we relied on 'Clue' on page 2. Some of you might have felt that that is a little out of nowhere. You can actually do it without resorting to Formula. For example, to find

$$(10) \qquad 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 = ?$$

just do

$$\begin{array}{ccccc} \binom{11}{2} & & \binom{12}{3} & & \binom{13}{4} \\ \| & & \| & & \| \\ 55 & & 220 & & 715 \\ {}_{11}\backslash & / {}_{-1} & {}_{12}\backslash & /{}_{-2} & \\ & 385 & & 1210 & \\ & {}_{11}\backslash & /{}_{-1} & & \\ & & 3025 & & \end{array}$$

Here, the calculation is

$$\begin{aligned} 11 \cdot 55 - 1 \cdot 220 &= 385, \\ 12 \cdot 220 - 2 \cdot 715 &= 1210, \qquad \text{and} \\ 11 \cdot 385 - 1 \cdot 1210 &= 3025. \end{aligned}$$

So the answer is 3025:

$$(10) \qquad 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 = 3025.$$

Similarly, to find

$$(20) \qquad 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 + 11^3 + 12^3 + 13^3 + 14^3 + 15^3 + 16^3 + 17^3 + 18^3 + 19^3 + 20^3 = ?$$

just do

$$\begin{array}{ccccc} \binom{21}{2} & & \binom{22}{3} & & \binom{23}{4} \\ \| & & \| & & \| \\ 210 & & 1540 & & 8855 \\ {}_{21}\diagdown & \diagup_{-1} & {}_{22}\diagdown & \diagup_{-2} & \\ & 2870 & & 16170 & \\ & {}_{21}\diagdown & \diagup_{-1} & & \\ & & 44100 & & \end{array}$$

Here, the calculation is

$$\begin{aligned} 21 \cdot 210 - 1 \cdot 1540 &= 2870, \\ 22 \cdot 1540 - 2 \cdot 8855 &= 16170, \qquad \text{and} \\ 21 \cdot 2870 - 1 \cdot 16170 &= 44100. \end{aligned}$$

So the answer is 44100:

$$(20) \qquad 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 + 11^3 + 12^3 + 13^3 + 14^3 + 15^3 + 16^3 + 17^3 + 18^3 + 19^3 + 20^3 = 44100.$$