Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XX

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Line #: 52920.

§20. Sum of consecutive square numbers.

• Today I want to do

(1) $1^2 = ?$

(2) $1^2 + 2^2 = ?$

(3) $1^2 + 2^2 + 3^2 = ?$

(4) $1^2 + 2^2 + 3^2 + 4^2 = ?$

(5) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = ?$

(6) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = ?$

(7) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = ?$

(8) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 = ?$

(9) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 = ?$

This resembles what you have seen before, though not exactly the same. So you at least have some ideas where this is going. Yes, we try to get hold of the patterns, and ultimately a formula. By now hopefully you know this is "not for nothing". People sometimes prejudge and undervalue the role of pattern recognitions in math. If you say pattern recognitions do not help your math skills, just hold on to that thought. True, you can just randomly make up pattern recognition problems, and those are just puzzles. Most of the puzzles are mathematically meaningless. But I also know that those people want to see some randomly created drills. For that, I say between randomly created puzzles and randomly created drills there is little difference, though I'm not saying I undervalue drills — in fact, I give more than a fair dose of drills in this class.

Now, I want to stress that in math some types of pattern recognitions prove to be meaningful, they are more than just puzzles. Here, 'meaningful' is a subjective word, but mathematicians have a rather clear idea what is meaningful and what is not. It is decided primarily by how much impact it has in a broader spectrum of math. The above pattern recognition problem does have great impacts. Indeed, we are going to rely on this to build some material, and then there is something else on top of it, building which we need to rely on that material, and so on so forth. In other words, the above problem is a piece that serves as a building block for the second half of the semester.

So, here is the deal. Let's do part (9). Now, you can certainly do it brute-force, but we are benefited by devising a method that works for other ones as well, as in part (20), part (100), part (1000), and so on. That's our goal. For that matter, it is good to recall (from "Review of Lectures – II"):

$$
\begin{aligned}
1 &= 1,\\
1+2 &= 3,\\
1+2+3 &= 6,\\
1+2+3+4 &= 10,\\
1+2+3+4+5 &= 15,\\
1+2+3+4+5+6 &= 21,\\
1+2+3+4+5+6+7 &= 28,\\
1+2+3+4+5+6+7+8 &= 36,\\
1+2+3+4+5+6+7+8+9 &= 45,\\
&\vdots \qquad \ddots\\
1+2+3+4+5+6+7+8+9+\cdots+n &= \frac{1}{2}\,n\left(n+1\right) = \binom{n+1}{2}
\end{aligned}
$$

(where the last line is the formula we pulled). Agree that, in this latter list, there is no square on each term. Now, remember that we used these answers and constructed

$$
\begin{aligned}
1 &= 1,\\
1+3 &= 4,\\
1+3+6 &= 10,\\
1+3+6+10 &= 20,\\
1+3+6+10+15 &= 35,\\
1+3+6+10+15+21 &= 56,\\
1+3+6+10+15+21+28 &= 84,\\
1+3+6+10+15+21+28+36 &= 120,\\
1+3+6+10+15+21+28+36+45 &= 165.
\end{aligned}
$$

Here, each number showing up in each line is a binomial coefficient. More precisely:

$$
\begin{aligned}
&\binom{2}{2} = 1 = \binom{3}{3},\\
&\binom{2}{2} + \binom{3}{2} = 4 = \binom{4}{3},\\
&\binom{2}{2} + \binom{3}{2} + \binom{4}{2} = 10 = \binom{5}{3},\\
&\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = 20 = \binom{6}{3},\\
&\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} = 35 = \binom{7}{3},\\
&\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} = 56 = \binom{8}{3},\\
&\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} + \binom{8}{2} = 84 = \binom{9}{3},\\
&\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} + \binom{8}{2} + \binom{9}{2} = 120 = \binom{10}{3},\\
&\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} + \binom{8}{2} + \binom{9}{2} + \binom{10}{2} = 165 = \binom{11}{3}.
\end{aligned}
$$

The fact that in each line the sum on the left-hand side indeed equals the right-most binomial coefficient is due to the Pascal. We may write these as

$$\frac{1\cdot 2}{2} = 1,$$
$$\frac{1\cdot 2}{2}+\frac{2\cdot 3}{2} = 4,$$
$$\frac{1\cdot 2}{2}+\frac{2\cdot 3}{2}+\frac{3\cdot 4}{2} = 10,$$
$$\frac{1\cdot 2}{2}+\frac{2\cdot 3}{2}+\frac{3\cdot 4}{2}+\frac{4\cdot 5}{2} = 20,$$
$$\frac{1\cdot 2}{2}+\frac{2\cdot 3}{2}+\frac{3\cdot 4}{2}+\frac{4\cdot 5}{2}+\frac{5\cdot 6}{2} = 35,$$
$$\frac{1\cdot 2}{2}+\frac{2\cdot 3}{2}+\frac{3\cdot 4}{2}+\frac{4\cdot 5}{2}+\frac{5\cdot 6}{2}+\frac{6\cdot 7}{2} = 56,$$
$$\frac{1\cdot 2}{2}+\frac{2\cdot 3}{2}+\frac{3\cdot 4}{2}+\frac{4\cdot 5}{2}+\frac{5\cdot 6}{2}+\frac{6\cdot 7}{2}+\frac{7\cdot 8}{2} = 84,$$
$$\frac{1\cdot 2}{2}+\frac{2\cdot 3}{2}+\frac{3\cdot 4}{2}+\frac{4\cdot 5}{2}+\frac{5\cdot 6}{2}+\frac{6\cdot 7}{2}+\frac{7\cdot 8}{2}+\frac{8\cdot 9}{2} = 120,$$
$$\frac{1\cdot 2}{2}+\frac{2\cdot 3}{2}+\frac{3\cdot 4}{2}+\frac{4\cdot 5}{2}+\frac{5\cdot 6}{2}+\frac{6\cdot 7}{2}+\frac{7\cdot 8}{2}+\frac{8\cdot 9}{2}+\frac{9\cdot 10}{2} = 165.$$

Multiply 2 to the two sides in each line, to get rid of the denominators:

$$1\cdot 2 = 2 = 2\binom{3}{3},$$
$$1\cdot 2 + 2\cdot 3 = 8 = 2\binom{4}{3},$$
$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 = 20 = 2\binom{5}{3},$$
$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 = 40 = 2\binom{6}{3},$$
$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6 = 70 = 2\binom{7}{3},$$
$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6 + 6\cdot 7 = 112 = 2\binom{8}{3},$$
$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6 + 6\cdot 7 + 7\cdot 8 = 168 = 2\binom{9}{3},$$
$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6 + 6\cdot 7 + 7\cdot 8 + 8\cdot 9 = 240 = 2\binom{10}{3},$$
$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6 + 6\cdot 7 + 7\cdot 8 + 8\cdot 9 + 9\cdot 10 = 330 = 2\binom{11}{3}.$$

So the nineth line is

$$(\mathrm{A})_9 \quad 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6 + 6\cdot 7 + 7\cdot 8 + 8\cdot 9 + 9\cdot 10 = 330.$$

Meanwhile,

$$(\mathrm{B})_9 \quad 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

(from page 2). Subtraciton side by side $(\mathrm{A})_9 - (\mathrm{B})_9$:

$$\begin{array}{lccccccccccccccccccc}
 & \underbrace{1\cdot 2} & + & \underbrace{2\cdot 3} & + & \underbrace{3\cdot 4} & + & \underbrace{4\cdot 5} & + & \underbrace{5\cdot 6} & + & \underbrace{6\cdot 7} & + & \underbrace{7\cdot 8} & + & \underbrace{8\cdot 9} & + & \underbrace{9\cdot 10} & & \\
 & \| & & \| & & \| & & \| & & \| & & \| & & \| & & \| & & \| & & \\
(\mathrm{A})_9 & \boxed{2} & + & \boxed{6} & + & \boxed{12} & + & \boxed{20} & + & \boxed{30} & + & \boxed{42} & + & \boxed{56} & + & \boxed{72} & + & \boxed{90} & = & \boxed{330} \\
(\mathrm{B})_9 & \boxed{1} & + & \boxed{2} & + & \boxed{3} & + & \boxed{4} & + & \boxed{5} & + & \boxed{6} & + & \boxed{7} & + & \boxed{8} & + & \boxed{9} & = & \boxed{45} \\
-) & \hline
 & \boxed{1} & + & \boxed{4} & + & \boxed{9} & + & \boxed{16} & + & \boxed{25} & + & \boxed{36} & + & \boxed{49} & + & \boxed{64} & + & \boxed{81} & = & \boxed{285}
\end{array}$$

Just look at the last line. Let's duplicate:

$$\boxed{1} + \boxed{4} + \boxed{9} + \boxed{16} + \boxed{25} + \boxed{36} + \boxed{49} + \boxed{64} + \boxed{81} = \boxed{285}$$

Realize that each number inside the box on the left-hand side is a square number. So, the answer is found. Namely:

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 = 285.$$

Now, those of you who still say just adding up the numbers

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64 \text{ and } 81$$

is faster, how about

$$\begin{aligned}
(100)\quad & 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 \\
& + 11^2 + 12^2 + 13^2 + 14^2 + 15^2 + 16^2 + 17^2 + 18^2 + 19^2 + 20^2 \\
& + 21^2 + 22^2 + 23^2 + 24^2 + 25^2 + 26^2 + 27^2 + 28^2 + 29^2 + 30^2 \\
& + 31^2 + 32^2 + 33^2 + 34^2 + 35^2 + 36^2 + 37^2 + 38^2 + 39^2 + 40^2 \\
& + 41^2 + 42^2 + 43^2 + 44^2 + 45^2 + 46^2 + 47^2 + 48^2 + 49^2 + 50^2 \\
& + 51^2 + 52^2 + 53^2 + 54^2 + 55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 \\
& + 61^2 + 62^2 + 63^2 + 64^2 + 65^2 + 66^2 + 67^2 + 68^2 + 69^2 + 70^2 \\
& + 71^2 + 72^2 + 73^2 + 74^2 + 75^2 + 76^2 + 77^2 + 78^2 + 79^2 + 80^2 \\
& + 81^2 + 82^2 + 83^2 + 84^2 + 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2 \\
& + 91^2 + 92^2 + 93^2 + 94^2 + 95^2 + 96^2 + 97^2 + 98^2 + 99^2 + 100^2 \; =?
\end{aligned}$$

You are probably dissuaded from doing it brute-force. Now, we already know how to do it in a fast-and-furious way. Namely, this is $(\mathrm{A})_{100} - (\mathrm{B})_{100}$, where

$(\mathrm{A})_{100}$

$$\begin{aligned}
& 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6 + 6\cdot 7 + 7\cdot 8 + 8\cdot 9 + 9\cdot 10 + 10\cdot 11 \\
& +11\cdot 12 + 12\cdot 13 + 13\cdot 14 + 14\cdot 15 + 15\cdot 16 + 16\cdot 17 + 17\cdot 18 + 18\cdot 19 + 19\cdot 20 + 20\cdot 21 \\
& +21\cdot 22 + 22\cdot 23 + 23\cdot 24 + 24\cdot 25 + 25\cdot 26 + 26\cdot 27 + 27\cdot 28 + 28\cdot 29 + 29\cdot 30 + 30\cdot 31 \\
& +31\cdot 32 + 32\cdot 33 + 33\cdot 34 + 34\cdot 35 + 35\cdot 36 + 36\cdot 37 + 37\cdot 38 + 38\cdot 39 + 39\cdot 40 + 40\cdot 41 \\
& +41\cdot 42 + 42\cdot 43 + 43\cdot 44 + 44\cdot 45 + 45\cdot 46 + 46\cdot 47 + 47\cdot 48 + 48\cdot 49 + 49\cdot 50 + 50\cdot 51 \\
& +51\cdot 52 + 52\cdot 53 + 53\cdot 54 + 54\cdot 55 + 55\cdot 56 + 56\cdot 57 + 57\cdot 58 + 58\cdot 59 + 59\cdot 60 + 60\cdot 61 \\
& +61\cdot 62 + 62\cdot 63 + 63\cdot 64 + 64\cdot 65 + 65\cdot 66 + 66\cdot 67 + 67\cdot 68 + 68\cdot 69 + 69\cdot 70 + 70\cdot 71 \\
& +71\cdot 72 + 72\cdot 73 + 73\cdot 74 + 74\cdot 75 + 75\cdot 76 + 76\cdot 77 + 77\cdot 78 + 78\cdot 79 + 79\cdot 80 + 80\cdot 81 \\
& +81\cdot 82 + 82\cdot 83 + 83\cdot 84 + 84\cdot 85 + 85\cdot 86 + 86\cdot 87 + 87\cdot 88 + 88\cdot 89 + 89\cdot 90 + 90\cdot 91 \\
& +91\cdot 92 + 92\cdot 93 + 93\cdot 94 + 94\cdot 95 + 95\cdot 96 + 96\cdot 97 + 97\cdot 98 + 98\cdot 99 + 99\cdot 100 \\
& \qquad +100\cdot 101
\end{aligned}$$

which is just 2 times $\binom{102}{3}$ (which can be easily seen once you go back to the bottom half of page 4), and

$$\begin{array}{l}(B)_{100} \quad 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\ \quad + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 \\ \quad + 21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 \\ \quad + 31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39 + 40 \\ \quad + 41 + 42 + 43 + 44 + 45 + 46 + 47 + 48 + 49 + 50 \\ \quad + 51 + 52 + 53 + 54 + 55 + 56 + 57 + 58 + 59 + 60 \\ \quad + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70 \\ \quad + 71 + 72 + 73 + 74 + 75 + 76 + 77 + 78 + 79 + 80 \\ \quad + 81 + 82 + 83 + 84 + 85 + 86 + 87 + 88 + 89 + 90 \\ \quad + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 + 100,\end{array}$$

which is indeed $\binom{101}{2} = 5050$ as found in "Review of Lectures – II". So, (100) is calculated simply as

$$(A)_{100} - (B)_{100} = 2 \cdot \binom{102}{3} - \binom{101}{2}.$$

To perform:

$$\begin{aligned} 2 \cdot \binom{102}{3} - \binom{101}{2} &= 2 \cdot \frac{100 \cdot 101 \cdot 102}{3 \cdot 2} - \frac{100 \cdot 101}{2} \\ &= 100 \cdot 101 \cdot \frac{102}{3} - 100 \cdot 101 \cdot \frac{1}{2} \\ &= 100 \cdot 101 \cdot 34 - 100 \cdot 101 \cdot \frac{1}{2} \\ &= 100 \cdot 101 \cdot \left(34 - \frac{1}{2}\right) \\ &= 10100 \cdot \frac{67}{2} \\ &= 5050 \cdot 67 = 338350. \end{aligned}$$

- More generally:

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = 2\binom{n+2}{3} - \binom{n+1}{2}.$$

Now,

$$\begin{aligned}
2\binom{n+2}{3} - \binom{n+1}{2} &= 2 \cdot \frac{n\left(n+1\right)\left(n+2\right)}{3 \cdot 2} - \frac{n\left(n+1\right)}{2} \\
&= n\left(n+1\right) \cdot \frac{n+2}{3} - n\left(n+1\right) \cdot \frac{1}{2} \\
&= n\left(n+1\right) \cdot \left(\frac{n+2}{3} - \frac{1}{2} \right) \\
&= n\left(n+1\right) \cdot \left(\frac{2n+4}{6} - \frac{3}{6} \right) \\
&= n\left(n+1\right) \cdot \frac{2n+4-3}{6} \\
&= n\left(n+1\right) \cdot \frac{2n+1}{6} \\
&= \frac{n\left(n+1\right)\left(2n+1\right)}{6}.
\end{aligned}$$

It's worth highlighting it:

Formula. Let n be a positive integer. Then the sum of the squares of the first n consecutive integers is

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 \quad = \quad \frac{1}{6}\, n\Big(n+1\Big)\Big(2n+1\Big).$$

Now, in retrospect, performing (100) is just an application of this last formula. Namely,

$$\begin{aligned}
(100) \quad & 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 \\
&+ 11^2 + 12^2 + 13^2 + 14^2 + 15^2 + 16^2 + 17^2 + 18^2 + 19^2 + 20^2 \\
&+ 21^2 + 22^2 + 23^2 + 24^2 + 25^2 + 26^2 + 27^2 + 28^2 + 29^2 + 30^2 \\
&+ 31^2 + 32^2 + 33^2 + 34^2 + 35^2 + 36^2 + 37^2 + 38^2 + 39^2 + 40^2 \\
&+ 41^2 + 42^2 + 43^2 + 44^2 + 45^2 + 46^2 + 47^2 + 48^2 + 49^2 + 50^2 \\
&+ 51^2 + 52^2 + 53^2 + 54^2 + 55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 \\
&+ 61^2 + 62^2 + 63^2 + 64^2 + 65^2 + 66^2 + 67^2 + 68^2 + 69^2 + 70^2 \\
&+ 71^2 + 72^2 + 73^2 + 74^2 + 75^2 + 76^2 + 77^2 + 78^2 + 79^2 + 80^2 \\
&+ 81^2 + 82^2 + 83^2 + 84^2 + 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2 \\
&+ 91^2 + 92^2 + 93^2 + 94^2 + 95^2 + 96^2 + 97^2 + 98^2 + 99^2 + 100^2
\end{aligned}$$

$$= \frac{100 \cdot \Big(100+1\Big) \cdot \Big(2 \cdot 100+1\Big)}{6}$$

$$= \frac{100 \cdot 101 \cdot 201}{6}$$

$$= \frac{2030100}{6} = 338350.$$

Exercise 1. Use formula above to find

$$(15) \quad 1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2+11^2+12^2+13^2+14^2+15^2.$$

[**Solution**] By formula, this equals

$$\frac{15 \cdot \left(15+1\right) \cdot \left(2 \cdot 15+1\right)}{6} = \frac{15 \cdot 16 \cdot 31}{6}$$

$$= \frac{240 \cdot 31}{6}$$

$$= 40 \cdot 31 = 1240.$$

Exercise 2. How much does it make if you add up the squares of integers between 1 and 300?

[**Solution**] By formula, this equals

$$\frac{300 \cdot \left(300+1\right) \cdot \left(2 \cdot 300+1\right)}{6} = \frac{300 \cdot 301 \cdot 601}{6}$$

$$= 50 \cdot 301 \cdot 601 = 9045050.$$