Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – II

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§2. Sum of consecutive integers.

Just out of the blue, can you do

(1) $1 = ?$

(2) $1 + 2 = ?$

(3) $1 + 2 + 3 = ?$

(4) $1 + 2 + 3 + 4 = ?$

(5) $1 + 2 + 3 + 4 + 5 = ?$

(6) $1 + 2 + 3 + 4 + 5 + 6 = ?$

(7) $1 + 2 + 3 + 4 + 5 + 6 + 7 = ?$

(8) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = ?$

(9) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = ?$

(10) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = ?$

Don't laugh. Just trust me I know what I am doing, and please take this problem seriously. Also, don't say we should let computers handle it, though computers are indeed capable of doing these. The answers are

(1) 1,

(2) 3,

(3) 6,

(4) 10,

(5) 15,

(6) 21,

(7) 28,

(8) 36,

(9) 45,

(10) 55.

First of all, part (1) alone is meaningless. In math, a statement of type

“1 equals 1.”

is called a ‘tautology’. Spelling:

tautology.

This word is a technical term in math for something being self-evident, self-referential, or catch 22. I included part (1) because that might help us see some patterns, if there is any (we will figure it out).

Up to part (3), I bet this is an easy piece of cake for you.

As for part (3), you can answer $1+2+3$ equals 6 in a split second. The thought process occuring in your head is $1+2$ equals 3, and $3+3$ equals 6. Very good.

But what about part (4): $1+2+3+4$? would you re-do $1+2=3$ and then re-do $3+3=6$, and then finally do $6+4$? No, you don't have to go like that. If you keep track of your previous answers, then you can take advantage of it. So for part (4), you can utilize the answer for part (3), as in

$$
\begin{aligned}
1 + 2 + 3 + 4 &= \underbrace{\Big(1 + 2 + 3\Big)}_{\substack{\| \text{ (by part (3))} \\ 6}} + 4 \\
&= 6 + 4 \\
&= 10.
\end{aligned}
$$

In the same spirit, we can do part (5):

$$
\begin{aligned}
1 + 2 + 3 + 4 + 5 &= \underbrace{\Big(1 + 2 + 3 + 4\Big)}_{\substack{\| \text{ (by part (4))} \\ 10}} + 5 \\
&= 10 + 5 \\
&= 15.
\end{aligned}
$$

Likewise, here is part (6):

$$
\begin{aligned}
1 + 2 + 3 + 4 + 5 + 6 &= \underbrace{\Big(1 + 2 + 3 + 4 + 5\Big)}_{\substack{\| \text{ (by part (5))} \\ 15}} + 6 \\
&= 15 + 6 \\
&= 21.
\end{aligned}
$$

The rest should be easy, if you employ the same trick. It doesn't take more than three minutes to do all the ten parts (1) through (10). So, to highlight the answers:

(1) $1 = 1.$

(2) $1 + 2 = 3.$

(3) $1 + 2 + 3 = 6.$

(4) $1 + 2 + 3 + 4 = 10.$

(5) $1 + 2 + 3 + 4 + 5 = 15$

(6) $1 + 2 + 3 + 4 + 5 + 6 = 21.$

(7) $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28.$

(8) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36.$

(9) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45.$

(10) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55.$

But you probably know by now occasionally I throw some vexing problems. How about

(100)
$$\begin{array}{l} 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\ + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 \\ + 21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 \\ + 31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39 + 40 \\ + 41 + 42 + 43 + 44 + 45 + 46 + 47 + 48 + 49 + 50 \\ + 51 + 52 + 53 + 54 + 55 + 56 + 57 + 58 + 59 + 60 \\ + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70 \\ + 71 + 72 + 73 + 74 + 75 + 76 + 77 + 78 + 79 + 80 \\ + 81 + 82 + 83 + 84 + 85 + 86 + 87 + 88 + 89 + 90 \\ + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 + 100 = ? \end{array}$$

You definitely want to cheat, right? But is there a way to cheat? If someone says "you can cheat by taking advantage of part (9), and part (10)", then do you have any clue?

[**Method A**] part (100) = Box #1 + Box #2 :

$$
\begin{array}{l}
0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \\
+ 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 \\
+ 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 \\
+ 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 \\
+ 40 + 40 + 40 + 40 + 40 + 40 + 40 + 40 + 40 + 40 \\
+ 50 + 50 + 50 + 50 + 50 + 50 + 50 + 50 + 50 + 50 \\
+ 60 + 60 + 60 + 60 + 60 + 60 + 60 + 60 + 60 + 60 \\
+ 70 + 70 + 70 + 70 + 70 + 70 + 70 + 70 + 70 + 70 \\
+ 80 + 80 + 80 + 80 + 80 + 80 + 80 + 80 + 80 + 80 \\
+ 90 + 90 + 90 + 90 + 90 + 90 + 90 + 90 + 90 + 90
\end{array}
$$

Box #1

$$
\begin{array}{l}
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10
\end{array}
$$

Box #2

Let's calculate $\underline{\text{ Box \#1 }}$ and $\underline{\text{ Box \#2 }}$ each. This is easy. As for $\underline{\text{ Box \#1 }}$,

0	+ 0	+ 0	+ 0	+ 0	+ 0	+ 0	+ 0	+ 0	+ 0
+ 10	+ 10	+ 10	+ 10	+ 10	+ 10	+ 10	+ 10	+ 10	+ 10
+ 20	+ 20	+ 20	+ 20	+ 20	+ 20	+ 20	+ 20	+ 20	+ 20
+ 30	+ 30	+ 30	+ 30	+ 30	+ 30	+ 30	+ 30	+ 30	+ 30
+ 40	+ 40	+ 40	+ 40	+ 40	+ 40	+ 40	+ 40	+ 40	+ 40
+ 50	+ 50	+ 50	+ 50	+ 50	+ 50	+ 50	+ 50	+ 50	+ 50
+ 60	+ 60	+ 60	+ 60	+ 60	+ 60	+ 60	+ 60	+ 60	+ 60
+ 70	+ 70	+ 70	+ 70	+ 70	+ 70	+ 70	+ 70	+ 70	+ 70
+ 80	+ 80	+ 80	+ 80	+ 80	+ 80	+ 80	+ 80	+ 80	+ 80
+ 90	+ 90	+ 90	+ 90	+ 90	+ 90	+ 90	+ 90	+ 90	+ 90
↑	↑	↑	↑	↑	↑	↑	↑	↑	↑

ten identical boxes,
each equals

$$\boxed{10 + 20 + 30 + 40 + 50 + 60 + 70 + 80 + 90} = 450$$

(Why? That's thanks to part (9):

$$\boxed{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9} = 45 \text{)}$$

Since there are ten of those boxes, each equals 450, so

$$\underline{\text{ Box \#1 }} = 450 \cdot 10 = 4500.$$

But that's not the final answer. Indeed, remember, we have to do Box #2 as well:

	1 +	2 +	3 +	4 +	5 +	6 +	7 +	8 +	9 +	10
+	1 +	2 +	3 +	4 +	5 +	6 +	7 +	8 +	9 +	10
+	1 +	2 +	3 +	4 +	5 +	6 +	7 +	8 +	9 +	10
+	1 +	2 +	3 +	4 +	5 +	6 +	7 +	8 +	9 +	10
+	1 +	2 +	3 +	4 +	5 +	6 +	7 +	8 +	9 +	10
+	1 +	2 +	3 +	4 +	5 +	6 +	7 +	8 +	9 +	10
+	1 +	2 +	3 +	4 +	5 +	6 +	7 +	8 +	9 +	10
+	1 +	2 +	3 +	4 +	5 +	6 +	7 +	8 +	9 +	10
+	1 +	2 +	3 +	4 +	5 +	6 +	7 +	8 +	9 +	10
+	1 +	2 +	3 +	4 +	5 +	6 +	7 +	8 +	9 +	10

ten identical boxes, each equal 55

So

$$\underline{\text{Box \#2}} = 55 \cdot 10 = 550.$$

In sum,

$$\text{part (100)} = \underline{\text{Box \#1}} + \underline{\text{Box \#2}} = 4500 + 550 = 5050.$$

Phew. This was still elaborate. But it wasn't like it took forever. Remember, the first method we employed was doing it one-by-one. Had we stuck with that method, it would have taken like ten hours. But with this new way, it took a matter of ten minutes. By the way, this new way is technically not cheating. This way is absolutely legit. In math, you can do whatever you want as long as each step is correct and you can pull the final answer. And the fewer the steps are, the better. That's one characteristic of math. But that's not the end of the story. Here is even faster way to do it.

[**Method B**] Part (100) equals the following (boxes are at first meaningless):

$$
\begin{array}{rrrrr}
\boxed{1}+100 & +\ \boxed{2}+99 & +\ \boxed{3}+98 & +\ \boxed{4}+97 & +\ \boxed{5}+96 \\
+\ \boxed{6}+95 & +\ \boxed{7}+94 & +\ \boxed{8}+93 & +\ \boxed{9}+92 & +\ \boxed{10}+91 \\
+\ \boxed{11}+90 & +\ \boxed{12}+89 & +\ \boxed{13}+88 & +\ \boxed{14}+87 & +\ \boxed{15}+86 \\
+\ \boxed{16}+85 & +\ \boxed{17}+84 & +\ \boxed{18}+83 & +\ \boxed{19}+82 & +\ \boxed{20}+81 \\
+\ \boxed{21}+80 & +\ \boxed{22}+79 & +\ \boxed{23}+78 & +\ \boxed{24}+77 & +\ \boxed{25}+76 \\
+\ \boxed{26}+75 & +\ \boxed{27}+74 & +\ \boxed{28}+73 & +\ \boxed{29}+72 & +\ \boxed{30}+71 \\
+\ \boxed{31}+70 & +\ \boxed{32}+69 & +\ \boxed{33}+68 & +\ \boxed{34}+67 & +\ \boxed{35}+66 \\
+\ \boxed{36}+65 & +\ \boxed{37}+64 & +\ \boxed{38}+63 & +\ \boxed{39}+62 & +\ \boxed{40}+61 \\
+\ \boxed{41}+60 & +\ \boxed{42}+59 & +\ \boxed{43}+58 & +\ \boxed{44}+57 & +\ \boxed{45}+56 \\
+\ \boxed{46}+55 & +\ \boxed{47}+54 & +\ \boxed{48}+53 & +\ \boxed{49}+52 & +\ \boxed{50}+51
\end{array}
$$

If you just look at those boxed ones, read them off from-left-to-right on each line, that forms the normal integer sequence 1—50. Meanwhile, the rest are the 'reversed' sequence 51—100. So this makes up 1—100. Now, let's rearrange the boxes:

$$
\begin{array}{rrrrr}
\boxed{1+100} & +\ \boxed{2+99} & +\ \boxed{3+98} & +\ \boxed{4+97} & +\ \boxed{5+96} \\
+\ \boxed{6+95} & +\ \boxed{7+94} & +\ \boxed{8+93} & +\ \boxed{9+92} & +\ \boxed{10+91} \\
+\ \boxed{11+90} & +\ \boxed{12+89} & +\ \boxed{13+88} & +\ \boxed{14+87} & +\ \boxed{15+86} \\
+\ \boxed{16+85} & +\ \boxed{17+84} & +\ \boxed{18+83} & +\ \boxed{19+82} & +\ \boxed{20+81} \\
+\ \boxed{21+80} & +\ \boxed{22+79} & +\ \boxed{23+78} & +\ \boxed{24+77} & +\ \boxed{25+76} \\
+\ \boxed{26+75} & +\ \boxed{27+74} & +\ \boxed{28+73} & +\ \boxed{29+72} & +\ \boxed{30+71} \\
+\ \boxed{31+70} & +\ \boxed{32+69} & +\ \boxed{33+68} & +\ \boxed{34+67} & +\ \boxed{35+66} \\
+\ \boxed{36+65} & +\ \boxed{37+64} & +\ \boxed{38+63} & +\ \boxed{39+62} & +\ \boxed{40+61} \\
+\ \boxed{41+60} & +\ \boxed{42+59} & +\ \boxed{43+58} & +\ \boxed{44+57} & +\ \boxed{45+56} \\
+\ \boxed{46+55} & +\ \boxed{47+54} & +\ \boxed{48+53} & +\ \boxed{49+52} & +\ \boxed{50+51}
\end{array}
$$

There are fifty boxes, and each equals 101. So the total equals

$$101 \cdot 50 = 5050.$$

⋆ Now, in retrospect, we could have done it this way (Method B) for

(1) $1 =?$

(2) $1 + 2 =?$

(3) $1 + 2 + 3 =?$

(4) $1 + 2 + 3 + 4 =?$

(5) $1 + 2 + 3 + 4 + 5 =?$

(6) $1 + 2 + 3 + 4 + 5 + 6 =?$

(7) $1 + 2 + 3 + 4 + 5 + 6 + 7 =?$

(8) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 =?$

(9) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 =?$

(10) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 =?$

For example, as for part (10):

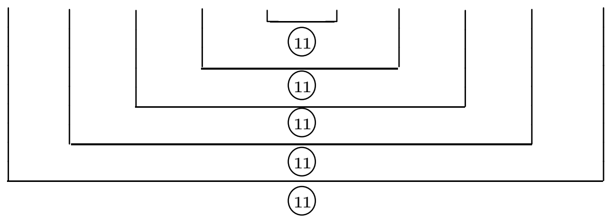
$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

$$= \underbrace{\boxed{1 + 10}}_{11} + \underbrace{\boxed{2 + 9}}_{11} + \underbrace{\boxed{3 + 8}}_{11} + \underbrace{\boxed{4 + 7}}_{11} + \underbrace{\boxed{5 + 6}}_{11}$$

$$= 11 \cdot 5 = 55.$$

Or, even without permuting numbers, you can do it like

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$



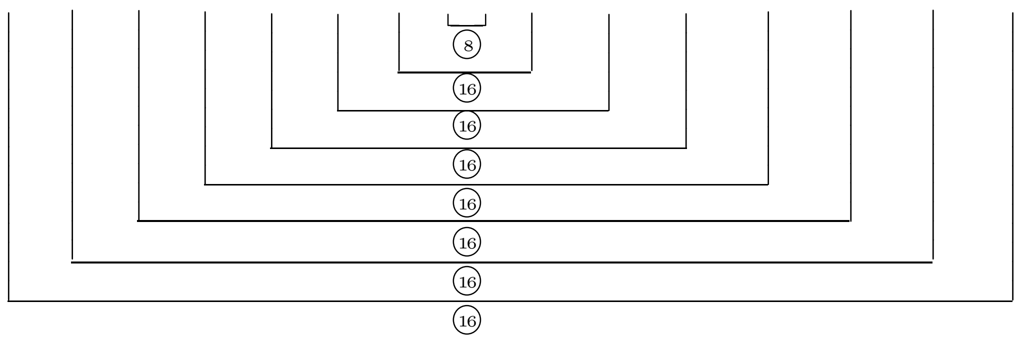
$$= 11 \cdot 5 = 55.$$

Exercise 1. Let's do

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15. \tag{15}$$

[**Solution**]

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15$$



$$= 8 + 16 \cdot 7$$

$$= 8 + 112$$

$$= 120.$$

⋆ So, in this exercise, not all numbers in the sequence 1—15 are paired, namely, 8 was left out. Can you explain why? That's clearly because 15 is an odd number.

Exercise 2.

(a) How much does it make if you add up integers between 1 and 20?

(b) How much does it make if you add up integers between 1 and 89?

(c) How much does it make if you add up integers between 1 and 163?

(d) How much does it make if you add up integers between 1 and 200?

But I bet you would certainly want to 'cheat' again. You would definitely want to use some formula so you wouldn't have to dissect the problem, find a clever arrangement of numbers, and then perform calculation each and every time. But does such a formula exist? The good news is, 'yes'.

Formula. Let n be a positive integer. Then the sum of the first n consecutive positive integers is given by the following:

$$\boxed{1 + 2 + 3 + 4 + \cdots + n \quad = \quad \frac{1}{2}\, n \left(n + 1\right).}$$

For example, part (a) of Exercise 2 above can be answered as follows:

[**Solution for (a)**]

$$\begin{aligned} &1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\ &+11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 \\ &\qquad = \frac{1}{2} \cdot 20 \left(20 + 1\right) \\ &\qquad = \frac{1}{2} \cdot 20 \cdot 21 \\ &\qquad = 10 \cdot 21 = 210. \end{aligned}$$