Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XIX

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§19. Logarithm.

- Today I introduce

"logarithm."

Logarithm is inseparably linked to the exponential functions. First, notations, and pronunciations:

$$\begin{array}{lcl}
\log_2 & : & \text{" log with base 2 ",} \\[2ex]
\log_3 & : & \text{" log with base 3 ",} \\[2ex]
\log_4 & : & \text{" log with base 4 ",} \\[2ex]
\log_5 & : & \text{" log with base 5 ",} \\[2ex]
\log_6 & : & \text{" log with base 6 ",} \\
\vdots & &
\end{array}$$

The base can be any positive real number (not necessarily an integer). Actually, from a logical standpoint, working with just one of those logs suffices, because all others are easily recovered from it. Though the choice of such 'representative log' is arbitrary, for a variety of reasons, the consensus most natural choice is the one with "base e":

$$\ln \quad = \quad \log_e \quad : \quad \text{"natural log".}$$

I will tell you the reason why. But for the time being I treat all of them equally.

The first thing you should know: The above are not numbers themselves. Each time you throw in a number, it becomes a number. In other words, they are functions. So, each of

$$\log_2, \quad \log_3, \quad \log_5, \quad \ln$$

is a function, whereas each of

$$\log_2 3, \quad \log_3 5, \quad \log_5 4, \quad \ln 6,$$

is a number, a real number, to be precise.

⋆ Now, before explaining the meaning of these numbers, let me give you the contexts. Where do these 'log's show up? I am going to tell you that first. This would serve as a motivation why we worry about log. Yes, let's go back to

$$2^x, \qquad e^x, \qquad 3^x, \qquad 4^x, \qquad \cdots \; .$$

At a first glance, these are seemingly unrelated. Indeed, if you are asked whether 2^x and 3^x are the same, or different, you would definitely say "different". Yes. So,

$$2^x \;\neq\; 3^x$$

as funcitons. (Here you might quibble that, for $x = 0$, the two sides are equal. True, for $x = 0$, we have $2^0 = 1$ and $3^0 = 1$. But 0 is the only value for x with which 2^x and 3^x are equal. For a non-zero real number x, 2^x and 3^x are always different. So, when I say "2^x and 3^x are different as functions", that is a correct statement.)

That said, 2^x and 3^x are actually related somehow. They are tweaks of each other. In what sense? Yes, it is in the following sense: There exists a constant (a real number) c such that

$$3^x \;=\; 2^{cx}.$$

Now, I bet you want me to disclose the undisclosed identity of that constant c.

Yes. It is actually

$$c = \log_2 3.$$

So,

$$3^x = 2^{(\log_2 3)\, x}.$$

Likewise,

$$4^x = 3^{(\log_3 4)\, x}.$$

$$5^x = 2^{(\log_2 5)\, x}.$$

$$e^x = 10^{(\log_{10} e)\, x}.$$

⋆ More generally:

$$\boxed{a^x = b^{(\log_b a)\, x}}\ .$$

Pop quiz. Can you fill in the boxes?

$$2^x = 6^{\boxed{}}.$$

$$10^x = e^{\boxed{}}.$$

[**Answers**]:

$$2^x = 6^{\boxed{(\log_6 2)\, x}}.$$

$$10^x = e^{\boxed{(\ln 10)\, x}}.$$

So, the first role of 'log' is it serves as a 'buffer', to go from one exponential function to another $\big($such as going from 2^x to $3^x\big)$. And, if you realize, this actually pretty much tells you what 'log's are.

Indeed, substitute $x = 1$ into

$$\boxed{a^x = b^{(\log_b a)\, x}}$$

and get

$$\boxed{a = b^{\log_b a}}\,.$$

So, the bottm line of what 'log' is is summarized in one line:

$$\text{“}\ \boxed{x = \log_b a}\ \ \underline{\underline{\textit{is a number satisfying}}}\ \ \boxed{b^x = a}\ \text{”}.$$

Pop quiz. Can you fill in the boxes?

$$2 = 6^{\boxed{}}\,.$$

$$10 = e^{\boxed{}}\,.$$

$\Big[$ <u>**Answers**</u> $\Big]$**:**

$$2 = 6^{\boxed{\log_6 2}}\,.$$

$$10 = e^{\boxed{\ln 10}}\,.$$

$\star$ There is one thing you might wonder at this stage:

- the relationship between $\log_2 3$ and $\log_3 2$,
- the relationship between $\log_3 4$ and $\log_4 3$,
- the relationship between $\log_7 5$ and $\log_5 7$,

and so on. This is simple:

$$\log_3 2 = \frac{1}{\log_2 3},$$

$$\log_3 4 = \frac{1}{\log_4 3},$$

$$\log_7 5 = \frac{1}{\log_5 7}.$$

More generally:

Fact. Let a and b be positive real numbers. Then

$$\boxed{\log_b a = \frac{1}{\log_a b}}\ .$$

Pop quiz. Write each of the following in the form

$$\log_{\square} \boxed{}\ .$$

(a) $\dfrac{1}{\log_3 11}$. (b) $\dfrac{1}{\log_{10} 24}$. (c) $\dfrac{1}{\ln 5}$.

[**Answers**]: (a) $\log_{11} 3$. (b) $\log_{24} 10$. (c) $\log_5 e$.

- **Special values of $\log_2 x$.**

So far we haven't discussed questions like how much is $\log_2 4$? Or how much is $\log_2 8$? For that matter, let's recall

$$\begin{aligned}
2^1 &= 2, \\
2^2 &= 4, \\
2^3 &= 8, \\
2^4 &= 16, \\
2^5 &= 32, \\
2^6 &= 64, \\
2^7 &= 128, \\
2^8 &= 256, \\
2^9 &= 512, \\
2^{10} &= 1024.
\end{aligned}$$

These translate into

$$\begin{aligned}
\log_2 2 &= 1, \\
\log_2 4 &= 2, \\
\log_2 8 &= 3, \\
\log_2 16 &= 4, \\
\log_2 32 &= 5, \\
\log_2 64 &= 6, \\
\log_2 128 &= 7, \\
\log_2 256 &= 8, \\
\log_2 512 &= 9, \\
\log_2 1024 &= 10.
\end{aligned}$$

Exercise 1. $\log_2 2048 \;=?$ $\quad \log_2 8192 \;=?$ $\quad \log_2 32768 \;=?$ $\quad \log_2 65536 \;=?$

Consult the table below, if necessary.

n	11	12	13	14	15	16
2^n	2048	4096	8192	16384	32768	65536

$\left[\underline{\textbf{Answers}}\right]$**:** $\log_2 2048 \;=\; 11.$ $\quad \log_2 28192 \;=\; 13.$

$\log_2 32668 \;=\; 15.$ $\quad \log_2 65536 \;=\; 16.$

Likewise,

$$
\begin{aligned}
2^{-1} &= \frac{1}{2}, \\
2^{-2} &= \frac{1}{4}, \\
2^{-3} &= \frac{1}{8}, \\
2^{-4} &= \frac{1}{16}, \\
2^{-5} &= \frac{1}{32}, \\
2^{-6} &= \frac{1}{64}, \\
2^{-7} &= \frac{1}{128}, \\
2^{-8} &= \frac{1}{256}, \\
2^{-9} &= \frac{1}{512}, \\
2^{-10} &= \frac{1}{1024}.
\end{aligned}
$$

These translate into

$$
\begin{aligned}
\log_2 \frac{1}{2} &= -1,\\
\log_2 \frac{1}{4} &= -2,\\
\log_2 \frac{1}{8} &= -3,\\
\log_2 \frac{1}{16} &= -4,\\
\log_2 \frac{1}{32} &= -5,\\
\log_2 \frac{1}{64} &= -6,\\
\log_2 \frac{1}{128} &= -7,\\
\log_2 \frac{1}{256} &= -8,\\
\log_2 \frac{1}{512} &= -9,\\
\log_2 \frac{1}{1024} &= -10.
\end{aligned}
$$

Exercise 2. $\log_2 \dfrac{1}{2048} = ?$ $\quad \log_2 \dfrac{1}{4096} = ?$ $\quad \log_2 \dfrac{1}{16384} = ?$

$\log_2 \dfrac{1}{65536} = ?$

(Consult the table in Exercise 1 above if necessary.)

[**Answers**]: $\log_2 \dfrac{1}{2048} = -11.$ $\quad \log_2 \dfrac{1}{4096} = -12.$

$\log_2 \dfrac{1}{16384} = -14.$ $\quad \log_2 \dfrac{1}{65536} = -16.$

- **Special values of $\log_3 x$.**

  Exactly the same deal for $\log_3$.

$$\begin{aligned} 3^1 &= 3, \\ 3^2 &= 9, \\ 3^3 &= 27, \\ 3^4 &= 81, \\ 3^5 &= 243, \\ 3^6 &= 729, \\ 3^7 &= 2187, \\ 3^8 &= 6561. \end{aligned}$$

These translate into

$$\begin{aligned} \log_3 3 &= 1, \\ \log_3 9 &= 2, \\ \log_3 27 &= 3, \\ \log_3 81 &= 4, \\ \log_3 243 &= 5, \\ \log_3 729 &= 6, \\ \log_3 2187 &= 7, \\ \log_3 6561 &= 8. \end{aligned}$$

Exercise 3. $\log_3 19683 \ =?$ $\log_3 177147 \ =?$ $\log_3 1594323 \ =?$

Consult the table below, if necessary.

n	9	10	11	12	13	14
3^n	19683	59049	177147	531441	1594323	4782969

$\left[\underline{\textbf{Answers}}\right]$**:** $\log_3 19683 \ = \ 9.$ $\log_3 177147 \ = \ 11.$

$$\log_3 1594323 \ = \ 13.$$

Likewise,

$$\begin{aligned}
3^{-1} &= \frac{1}{3}, \\
3^{-2} &= \frac{1}{9}, \\
3^{-3} &= \frac{1}{27}, \\
3^{-4} &= \frac{1}{81}, \\
3^{-5} &= \frac{1}{243}, \\
3^{-6} &= \frac{1}{729}, \\
3^{-7} &= \frac{1}{2187}, \\
3^{-8} &= \frac{1}{6561}.
\end{aligned}$$

These translate into

$$\log_3 \frac{1}{3} = -1,$$

$$\log_3 \frac{1}{9} = -2,$$

$$\log_3 \frac{1}{27} = -3,$$

$$\log_3 \frac{1}{81} = -4,$$

$$\log_3 \frac{1}{243} = -5,$$

$$\log_3 \frac{1}{729} = -6,$$

$$\log_3 \frac{1}{2187} = -7,$$

$$\log_3 \frac{1}{6561} = -8.$$

Exercise 4. $\log_3 \dfrac{1}{59049} = ?$ $\quad\log_3 \dfrac{1}{531441} = ?$

$\log_3 \dfrac{1}{4782969} = ?$

(Consult the table in Exercise 3 above if necessary.)

$\left[\underline{\textbf{Answers}}\right]$**:** $\log_3 \dfrac{1}{59049} = -10.$ $\quad\log_3 \dfrac{1}{531441} = -12.$

$\log_2 \dfrac{1}{4782969} = -14.$

- **Special values of $\log_{10} x$.**

  Now, the same deal for $\log_{10}$.

$$\begin{aligned}
10^1 &= 10, \\
10^2 &= 100, \\
10^3 &= 1000, \\
10^4 &= 10000, \\
10^5 &= 100000, \\
10^6 &= 1000000, \\
10^7 &= 10000000, \\
10^8 &= 100000000.
\end{aligned}$$

These translate into

$$\begin{aligned}
\log_{10} 10 &= 1, \\
\log_{10} 100 &= 2, \\
\log_{10} 1000 &= 3, \\
\log_{10} 10000 &= 4, \\
\log_{10} 100000 &= 5, \\
\log_{10} 1000000 &= 6, \\
\log_{10} 10000000 &= 7, \\
\log_{10} 100000000 &= 8.
\end{aligned}$$

Likewise,

$$\begin{aligned}
10^{-1} &= \frac{1}{10}, \\
10-2 &= \frac{1}{100}, \\
10-3 &= \frac{1}{1000}, \\
10-4 &= \frac{1}{10000}, \\
10-5 &= \frac{1}{100000}, \\
10-6 &= \frac{1}{1000000}, \\
10-7 &= \frac{1}{10000000}, \\
10-8 &= \frac{1}{100000000}.
\end{aligned}$$

These translate into

$$\begin{aligned}
\log_{10} \frac{1}{10} &= -1, \\
\log_{10} \frac{1}{100} &= -2, \\
\log_{10} \frac{1}{1000} &= -3, \\
\log_{10} \frac{1}{10000} &= -4, \\
\log_{10} \frac{1}{100000} &= -5, \\
\log_{10} \frac{1}{1000000} &= -6, \\
\log_{10} \frac{1}{10000000} &= -7, \\
\log_{10} \frac{1}{100000000} &= -8.
\end{aligned}$$

Exercise 5. $\log_{10} 1000000000 \ =?$ $\big($1 followed by nine straight 0s$\big)$.

$\log_{10} \dfrac{1}{1000000000000} \ =?$ $\big($The denominator: 1 followed by twelve straight 0s$\big)$.

$\log_{10} 10^{24} \ =?$ $\log_{10} 10^{-60} \ =?$

$\Big[$ **Answers** $\Big]$**:**

$$\log_{10} 1000000000 \ = \ 9.$$

$$\log_{10} \frac{1}{1000000000000} \ = \ -12.$$

$$\log_{10} 10^{24} \ = \ 24. \qquad \log_{10} 10^{-60} \ = \ -60.$$

As you may realize by now,

$$\boxed{\log_{10} 10^n \ = \ n}\ .$$

Also, in retrospect,

$$\boxed{\log_2 2^n \ = \ n}$$

$\big($page 6–8$\big)$, and

$$\boxed{\log_3 3^n \ = \ n}$$

$\big($page 9–11$\big)$. More generally:

$$\boxed{\log_a a^n \ = \ n}$$

$\big($where n is an integer$\big)$.

$\star$ What's more, in this there is no reason n has to be an integer in order for the statement to be true.

$$\boxed{\log_a a^x \;=\; x}$$

(where x is a real number).

- **$\log_b 1$.** No matter what b is (provided b is a positive real number and $b \neq 1$), $\log_b 1$ always equal to 0.

$$\boxed{\log_b 1 \;=\; 0.} \;.$$

- **$\log_1 a$ is undefined.**

$\log_1 a$ is undefined. This is just because 1^x always equals 1, no matter what x is. Another way to see it is $\log_1 a$ would be the reciprocal of $\log_a 1$, but $\log_a 1$ equals 0. The reciprocal of 0 is undefined. (You might quibble that, because of $1^1 = 1$, we should say $\log_1 1 = 1$. True. However, someone else might argue that, because of $1^0 = 1$, we should say $\log_1 1 = 0$. The bottom line is, $\log_1 a$ for $a \neq 1$ is undefined, so, considering $\log_1 a$ as a function on a is pointless.)

- **$\log_0 a$ is undefined.**

$\log_0 a$ is undefined. This is just because 0^x always equals 0, no matter what x is (provided x is positive).

- **$\log_b 0$ is undefined.**

$\log_b 0$ is undefined. Actually, depending on a context, provided b is a positive real number and $b \neq 1$, $\log_b 0$ makes sense as a limit

$$\lim_{x \longrightarrow 0} \log_b x.$$

Let's not worry about this for now, though just in case

$$\lim_{x \longrightarrow 0} \log_b x \quad = \quad \begin{cases} -\infty & \bigl(b > 1\bigr), \\[2ex] +\infty & \bigl(b < 1\bigr). \end{cases}$$

- So, from now on, when we talk about $\boxed{\log_a b}$, we always assume

$$\boxed{a > 0, \qquad a \neq 1, \qquad \text{and} \qquad b > 0.}$$

In the future, whenever we write $\log_a b$, these conditions on a and b will be automatically assumed.

- **Summary.**

This is a good place to review two important things we have learned so far. One:

" $\boxed{x = \log_b a}$ *is a number satisfying* $\boxed{b^x = a}$ ".

(This is from page 4.) In particular,

$$b^{\log_b a} = a \qquad \big(a > 0\big).$$

Two:

$$\log_a a^x = x \qquad \big(x \text{ is a real number}\big).$$

(This is from page 15.)

These are usually put together, and called cancellation laws:

- **Cancellation laws.**

$$\boxed{\begin{array}{lll} b^{\log_b a} = a & & \big(a > 0\big). \\ \log_a a^x = x & & \big(x \text{ is a real number}\big). \end{array}}$$

It is worthwhile to isolate the cancellation laws for the natural log 'ln':

- **Cancellation laws for 'ln'.**

$$\boxed{\begin{aligned} e^{\ln a} &= a \qquad \big(a > 0\big). \\ \ln e^{x} &= x \qquad \big(x \text{ is a real number}\big). \end{aligned}}$$

Exercise 6. Use cancellation laws to simplify:

(1) $2^{\log_2 5}$. (2) $3^{\log_3 10}$. (3) $5^{\log_5 \frac{7}{3}}$. (4) $e^{\ln \sqrt{2}}$.

(5) $9^{\log_3 5}$. $\Big($Hint: $9 = 3 \cdot 3$, so $9^{\log_3 5} = 3^{\log_3 5} \cdot 3^{\log_3 5}$.$\Big)$

$\Big[$**Answers**$\Big]$: (1) 5. (2) 10. (3) $\dfrac{7}{3}$. (4) $\sqrt{2}$.

(5) 25.

Exercise 7. Use cancellation laws to simplify:

(1) $\log_3 3^6$. (2) $\log_2 2^{\frac{7}{2}}$. (3) $\log_{10} \sqrt{10}$. (4) $\ln e^{\pi}$.

(5) $\log_{49} 7$. $\Big($Hint: $7 = 49^{\frac{1}{2}}$.$\Big)$

$\Big[$**Answers**$\Big]$: (1) 6. (2) $\dfrac{7}{2}$. (3) $\dfrac{1}{2}$. (4) π.

(5) $\dfrac{1}{2}$.

- **Change of base.**

There are some important laws about 'log'. Just like the two exponential functions are related, two logarithms are related:

$$\log_b\, c \;=\; \frac{\log_a\, c}{\log_a\, b}.$$

In particular,

$$\log_b\, c \;=\; \frac{\ln\, c}{\ln\, b}.$$

Exercise 8. Simplify:

(1) $\dfrac{\log_3\, 7}{\log_3\, 4}$. (2) $\dfrac{\log_{11}\, 26}{\log_{11}\, 15}$. (3) $\dfrac{\ln\, 100}{\ln\, 10}$. (4) $\dfrac{\log_2\, 7}{\log_2\, e}$.

Write the answer in the form

$$\log_{\square} \boxed{} \qquad \text{or} \qquad \ln \boxed{}$$

(which ever is applicable). If there is still a room for simplification, simplify.

[**Answers**]: (1) $\log_4 7$. (2) $\log_{15} 26$. (3) $\log_{10} 100 \;=\; 2$. (4) $\ln 7$.

$\star$ Now we are finally ready to get into the most important topic, an extremely important property the logarithm possesses, called 'the logarithmic laws'. Here, you remember that I said two things early on: (a) "working with just one log suffices", and (b) "the consensus most natural choice is 'ln', the natural log". The above 'change of base' rule explains the reason for (a). It still does not explain the reason for (b). I will ultimately convince you about (b), though not today. So you have to take my words for it. Namely, in math, most of the time we just stick with 'ln'. At least within the current knowledge you will agree that the 'logarithmic laws' specifically for 'ln' (starting next page) allows you to easily reconstruct the same laws for logs with different bases. For that reason, in what follows we just work with 'ln'.

- **Logarithmic Laws.**

Below (i), (ii) and (iii) are the logarithmic laws for 'ln'.

(i)
$$\ln\Big(x\,y\Big) \;=\; \Big(\ln x\Big) + \Big(\ln y\Big)$$
$$\Big(x > 0, \quad y > 0\Big),$$

(ii)
$$\ln\frac{x}{y} \;=\; \Big(\ln x\Big) - \Big(\ln y\Big)$$
$$\Big(x > 0, \quad y > 0\Big),$$

(iii)
$$\ln\Big(x^{a}\Big) \;=\; a\Big(\ln x\Big)$$
$$\Big(x > 0\Big).$$

$\star$ Though there is no compelling reason to do so, just for once I want to use the symbols $\heartsuit$, $\diamondsuit$, $\clubsuit$ for x, y and z. It will give you a different impression:

(i)
$$\ln\Big(\heartsuit\,\diamondsuit\Big) \;=\; \Big(\ln\heartsuit\Big) + \Big(\ln\diamondsuit\Big)$$
$$\Big(\heartsuit > 0, \quad \diamondsuit > 0\Big),$$

(ii)
$$\ln\frac{\heartsuit}{\diamondsuit} \;=\; \Big(\ln\heartsuit\Big) - \Big(\ln\diamondsuit\Big)$$
$$\Big(\heartsuit > 0, \quad \diamondsuit > 0\Big),$$

(iii)
$$\ln\Big(\heartsuit^{\clubsuit}\Big) \;=\; \clubsuit\Big(\ln\heartsuit\Big)$$
$$\Big(\heartsuit > 0\Big).$$

- Let's isolate the case $\clubsuit = \dfrac{1}{2}$ in (iii):

$$\ln\Big(\heartsuit^{\frac{1}{2}}\Big) = \frac{1}{2}\Big(\ln \heartsuit\Big) \qquad\qquad \Big(\heartsuit > 0\Big).$$

or the same

$$\ln\sqrt{\heartsuit} = \frac{1}{2}\Big(\ln \heartsuit\Big) \qquad\qquad \Big(\heartsuit > 0\Big).$$

Example. $\Big(\ln 2\Big) + \Big(\ln 3\Big)$ equals $\ln 6.$ Note that

$$\Big(\ln 2\Big) + \Big(\ln 3\Big) \neq \ln 5.$$

Example. $\Big(\ln 3\Big) - \Big(\ln 2\Big)$ equals $\ln \dfrac{3}{2}.$ Note that

$$\Big(\ln 3\Big) - \Big(\ln 2\Big) \neq \ln 1.$$

Example. $5\ \ln 2$ equals $\ln 32.$ Note that

$$5\ \ln 2 \neq \ln 10.$$

Example. $\dfrac{1}{2}\ \ln 6$ equals $\ln\sqrt{6}\,.$ Note that

$$\frac{1}{2}\ \ln 6 \neq \ln 3.$$

Example. Let's simplify

$$e^{(\ln 3) + (\ln 7)}.$$

We use (i) of the Logarithmic Laws, and Cancellation Laws. This quantity is of the form $e^{\heartsuit}$, where $\heartsuit = \Big(\ln 3 \Big) + \Big(\ln 7 \Big)$. By (i) of the Logarithmic Laws, this $\heartsuit$ equals $\ln 21$. Hence the original quantity $e^{\heartsuit}$ equals $e^{\ln 21}$. By Cancellation Laws, this quantity equals 21.

Example. Let's simplify

$$e^{3\ \ln\ 2}.$$

We use (iii) of the Logarithmic Laws, and Cancellation Laws. This quantity is of the form $e^{\heartsuit}$, where $\heartsuit = 3\ln 2$. By (iii) of the Logarithmic Laws, this $\heartsuit$ equals $\ln 8$. Hence the original quantity $e^{\heartsuit}$ equals $e^{\ln 8}$. By Cancellation Laws, this quantity equals 8.

Example. Let's simplify

$$\ln\ 2^{\frac{1}{\ln\ 2}}.$$

We use (iii) of the Logarithmic Laws. First, the above quantity is of the form $\ln\ \heartsuit^{\clubsuit}$ where $\heartsuit = 2$, and $\clubsuit = \dfrac{1}{\ln 2}$. By (iii) of the Logarithmic Laws, this quantity equals $\clubsuit\ \ln \heartsuit$, that is,

$$\frac{1}{\ln 2}\ \cdot\ \ln 2.$$

This is simplified to 1. So the answer is 1.

Example. Let's simplify

$$2^{\frac{1}{\ln\ 2}}.$$

As for this, we have just worked out in the previous example that, ' ln ' of the quantity $2^{\frac{1}{\ln\ 2}}$ equals 1. Hence the quantity $2^{\frac{1}{\ln\ 2}}$ itself equals e. So, the answer is e.

Exercise 9.

(1) Simplify $\Big(\ln 3\Big) + \Big(\ln 12\Big)$. Write your answer in the form $\ln \boxed{}$.

(2) Simplify $\Big(\ln 15\Big) - \Big(\ln 5\Big)$. Write your answer in the form $\ln \boxed{}$.

(3) Rewrite $4 \ln 3$ in the form $\ln \boxed{}$.

(4) Rewrite $\ln 256$ in the form $\boxed{}\Big(\ln 2\Big)$.

(5) Rewrite $\ln \sqrt[3]{7}$ in the form $\dfrac{1}{\boxed{}}\Big(\ln 7\Big)$.

(6) Rewrite $\ln \sqrt[5]{81}$ in the form $\dfrac{\boxed{}}{\boxed{}}\Big(\ln 3\Big)$.

(7) Simplify $e^{(\ln 4)+(\ln 13)}$.

(8) Simplify $e^{2\,(\ln 5)}$.

(9) Simplify $\ln 5^{\frac{1}{\ln 5}}$.

(10) Simplify $5^{\frac{1}{\ln 5}}$.

$\Big[$**Answers**$\Big]$: (1) $\ln 36$. (2) $\ln 3$. (3) $\ln 81$. (4) $8 \ln 2$.

(5) $\dfrac{1}{3} \ln 7$. (6) $\dfrac{4}{5} \ln 3$. (7) 52. (8) 25. (9) 1.

(10) e.

Exercise 10. Verify:

(1) $2^{\frac{x}{\ln 2}} \;=\; e^x.$

(2) $x^x \;=\; e^{x\,\ln x}.$

(3) $x^{\frac{1}{x}} \;=\; e^{\frac{\ln x}{x}}.$

(4) $2^x \;=\; e^{x\,(\ln 2)}.$

(5) $x^{\ln x} \;=\; e^{\left((\ln x)^2\right)}.$

(6) $2^{\ln x} \;=\; e^{(\ln 2)(\ln x)} \;=\; x^{\ln 2}.$

(7) $\left(\ln x\right)^{\frac{1}{x}} \;=\; e^{\frac{\ln(\ln x)}{x}}.$

$\left[\underline{\textbf{Solutions}}\right]$**:** Take 'ln' of the two sides, and verify that the resulting qualtities from the two sides are equal.

Exercise 11.

(1) Are $x^{\left(y^z\right)}$ and $\left(x^y\right)^z$ the same?

(2) Are $\left(x^y\right)^{(z^w)}$, $x^{\left(\left(y^z\right)^w\right)}$ and $x^{\left(y^{\left(z^w\right)}\right)}$ all the same?

$\left[\underline{\textbf{Answers}}\right]$**:** (1) They are different. Indeed,

$$2^{\left(3^2\right)} = 2^9, \qquad \left(2^3\right)^2 = 8^2 = 2^6.$$

(2) They are all different. Indeed,

$$\left(2^3\right)^{(2^3)} = 8^8 = 2^{24}, \qquad 2^{\left(\left(3^2\right)^3\right)} = 2^{\left(9^3\right)} = 2^{729},$$

$$2^{\left(3^{\left(2^3\right)}\right)} = 2^{\left(3^8\right)} = 2^{6561}.$$