Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – XVIII

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§18. Exponential functions.

- Today we talk about

$$2^x, \qquad 3^x, \qquad 4^x, \quad \cdots$$

as functions on x. The theoretical stumbling block was how you raise the power of an irrational number. Last time we have managed to define $2^{\sqrt{2}}$. The general case of

$$\boxed{a^x \qquad \Big(a \ : \ \text{a positive real number}; \quad x \ : \ \text{a real number}\Big)}$$

can be easily extrapolated. Namely, we make the following definition:

Definition. Assume that a is a positive real number, x is a real number, and ℓ is an integer, $\ell > 1$. For each index n, let $c_n\big(\ell\big)$ be the truncation at the n-th digit under the "ℓ-ary" point of x. Define

$$\boxed{a^x \quad = \quad \lim_{n \to \infty} a^{c_n(\ell)}} \ .$$

This limit exists, and it does not depend on the choice of ℓ.

- **Coincidence of two definitions when x is rational.**

Realize that, when x is a rational number, then you have two definitions for a^x: One is the original definition, that is, write x as $\dfrac{n}{k}$ and a^x is the k-th root of a^n. Another is the limit of $a^{c_n(\ell)}$ as above, where $a^{c_n(\ell)}$ resorts to the original definition $\big($which makes sense because $c_n(\ell)$ is a rational number.$\big)$ For example, consider

$$x = \frac{1}{3}.$$

Then, in one definition a^x is the cube-root of x. On the other hand, in the usual decimals x is

$$x = 0.33333333333333333... \ ,$$

So $c_n\big(10\big)$ is like

$$\begin{aligned} c_1\big(10\big) &= 0.3 \ , \\ c_2\big(10\big) &= 0.33 \ , \\ c_3\big(10\big) &= 0.333 \ , \\ c_4\big(10\big) &= 0.3333 \ , \\ c_5\big(10\big) &= 0.33333 \ , \\ c_6\big(10\big) &= 0.333333 \ , \\ c_7\big(10\big) &= 0.3333333 \ , \\ c_8\big(10\big) &= 0.33333333 \ , \\ &\vdots \end{aligned}$$

None of these equals x. So it remains to be seen if the limit

$$\lim_{n \longrightarrow \infty} a^{c_n(\ell)}$$

indeed matches $a^{\frac{1}{3}}$ in the usual sense of the cube-root.

Now, the answer is 'yes indeed'. The easiest way to see this is as follows: x is re-expressed as

$$x = 0.1$$

in the base $\ell = 3$ system $\big($or, the ternary system$\big)$. Then, for $\ell = 3$, all $c_n\big(3\big)$ are actually equal to x. Then clearly $a^{c_n(3)}$ are all equal to a^x where x is the rational number $\frac{1}{3}$. So the limit

$$\lim_{n \to \infty} a^{c_n(3)}$$

is trivially equal to the cube-root of a. Now, we have already proved in our last lecture that changing ℓ does not affect the limit

$$\lim_{n \to \infty} a^{c_n(\ell)}.$$

So no matter which ℓ you choose, you have the same answer, namely, the limit

$$\lim_{n \to \infty} a^{c_n(\ell)}$$

always equals the cube-root of a.

As for the general case, you resort to the following fact:

Proposition. Let x be an arbitrary rational number. Then there exists ℓ such that the ℓ-ary expression of x has only finitely many digits.

This proposition is not difficult to prove — almost trivial to mathematicians. If this is trivial to you, then you think like a mathematician. If this is not trivial to you, spend some time to think about it.

- **Monotonicity.**

So far throughout I have relied on the following:

Proposition. Let a be a real number. Assume $a > 1$. Let r and s be rational numbers. Suppose $r < s$. Then $a^r < a^s$.

$\star$ Now that we have defined a^x when x is a real number, so it makes sense to ask if the same statement as Proposition above remains true even if we relaxed the assumption r and s are rational numbers. The answer is actually affirmative. Below let's change the letter from r and s to x and y.

Proposition refined. Let a be a real number. Assume $a > 1$. Let x and y be <u>real</u> numbers. Suppose $x < y$. Then $a^x < a^y$.

- **Continuity of a^x as a function on x.**

Next, continuity. Let me duplicate the theorem which we have covered in "Review of Lectures – XV" (Theorem 1 on page 19):

Theorem. Let r be a positive rational number, and fixed. Let a be a positive real number (not necessarily a rational number), and fixed. Then the following conclusion holds:

No matter how large an integer $N \, (> 0)$ you choose, you can find a rational number s (i) above r and (ii) below r each, such that the distance between a^r and a^s is less than $\dfrac{1}{N}$.

$\star$ Once again, at that time, we have not had the definition of a^x when x is irrational. It makes sense to ask if the same statement as Theorem above remains true even if we relaxed the assumption r is a rational number. The answer is, once again, affirmative. Below let's change the letter from r and s to x and y.

Theorem refined. Let x be a positive <u>real</u> number, and fixed. Let a be a positive real number, and fixed. Then the following conclusion holds:

No matter how large an integer $N \, (> 0)$ you choose, you can find a real number y (i) above x and (ii) below x each, such that the distance between a^x and a^y is less than $\dfrac{1}{N}$.

- If you are super-meticulous, you would raise the question, that, while you can see that (ii) is always true, it remains to be seen whether (i) is true, when x is irrational. You don't see if it can be proved strictly within what's covered in "Review of Lectures – XVII". Indeed, the definition of a^x uses $\Big\{ c_n \, (\ell) \Big\}_n$ which is a monotonially <u>increasing</u> sequence converging to x. You need to prove the fact that, if you have two sequences, one monotonially <u>increasing</u>, and one monotonially <u>decreasing</u>, both coverging to x, call them $\Big\{ c_n \Big\}_n$, and $\Big\{ d_n \Big\}_n$, respectively, then the limit of a^{c_n} and the limit of a^{d_n} coincide. Proof of (ii) would hinge on that. If you say so, you are absolutely correct. I can prove all these, but that is rather techinical, so I will just omit it.

• In the same spirit, it makes sense to ask if the exponential laws (from "Review of Lectures – XVI", page 11) remain valid even if we relax the assumption the exponents are rational. The answer to this question too is affirmative. Let me highlight:

Exponential Laws (refined). Let x and y be real numbers. Let a and b be positive real numbers Then

Rule I. $$\boxed{\left(ab\right)^{x} = a^{x}\, b^{x}}\,.$$

Rule II. $$\boxed{a^{x}\, a^{y} = a^{x+y}}\,.$$

Rule III. $$\boxed{\left(a^{x}\right)^{y} = a^{xy}}\,.$$

Rule IV. $$\boxed{a^{0} = 1}\,, \qquad \boxed{1^{x} = 1}\,.$$

Rule V. $$\boxed{a^{-x} = \frac{1}{a^{x}}}\,.$$

Exercise 1.

(1) Simplify $2^{x} \cdot 5^{x}$. Write your answer as in $\boxed{}^{x}$.

(2) Simplify $a^{3} \cdot a^{8}$.

(3) Simplify $\left(a^{\sqrt{2}}\right)^{\sqrt{2}}$.

(4) Simplify $1^{\sqrt{3}}$.

(5) Rewrite $a^{-\sqrt{5}}$ in the form $\dfrac{1}{\boxed{}}$.

[**Answers**]: (1) 10^{x}. (2) a^{11}. (3) a^{2}. (4) 1. (5) $\dfrac{1}{a^{\sqrt{5}}}$.

- $\boldsymbol{e^x}$**.**

Now, among all exponential functions a^x, the one with $a = e$ has a very very special place. Often when we say "the exponential function", it refers to e^x. Here is the reason why:

Theorem. Let x be an arbitrary real number. Then

$$\boxed{\begin{aligned} e^x &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\ &= \lim_{k\to\infty}\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\cdots+\frac{1}{k!}x^k\right). \end{aligned}}$$

- **Notational remark.** We often write

$$\lim_{k\to\infty}\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\cdots+\frac{1}{k!}x^k\right)$$

as

$$1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\cdots \quad .$$

If you incorporate this notation, then the above theorem is paraphrased as follows:

Theorem paraphrased. Let x be an arbitrary real number. Then

$$\boxed{\begin{aligned} e^x &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\ &= 1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\cdots \end{aligned}}$$

Example 1.
$$\sqrt{e} = 1 + \frac{1}{1!}\cdot\frac{1}{2} + \frac{1}{2!}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{3!}\cdot\left(\frac{1}{2}\right)^3 + \frac{1}{4!}\cdot\left(\frac{1}{2}\right)^4 + \frac{1}{5!}\cdot\left(\frac{1}{2}\right)^5 + \frac{1}{6!}\cdot\left(\frac{1}{2}\right)^6 + \cdots .$$

• Can you use your calculator, to calculate the right-hand side (say, the first ten terms) and then independently of that, do $\sqrt{e}$, and see if the two results match?

• **Exponential Laws pertaining to e^x.**

Rule II. $$\boxed{e^x\, e^y = e^{x+y}} .$$

Rule III. $$\boxed{\left(e^x\right)^y = e^{xy}} .$$

Rule IV. $$\boxed{e^0 = 1} .$$

Rule V. $$\boxed{e^{-x} = \frac{1}{e^x}} .$$

• Now, in the next lecture we introduce 'logarithm'. Logarithm and exponential functions are inseparably linked. You cannot talk about one without referring to the other. Stated in other words, the exponential functions (including e^x) can be best understood within the framework of 'logarithm'. So let's look forward to the next lecture.

Exercise 2. Find the limits:

(1) $$\lim_{n \to \infty} \left(1+\frac{3}{n}\right)^n = ?$$ (2) $$\lim_{n \to \infty} \left(1-\frac{1}{n}\right)^n = ?$$

(3) $$\lim_{n \to \infty} \left(1-\frac{\sqrt{2}}{n}\right)^n = ?$$

$\left[\underline{\textbf{Answers}}\right]$**:** (1) e^3. (3) e^{-1}. (2) $e^{-\sqrt{2}}$.

Exercise 3. Write up each of (1) e^2, (1) $\sqrt[3]{e}$, and (3) e^{-1} as an infinite sum in the same fashion as Example 1.

$\left[\underline{\textbf{Answers}}\right]$**:**

(1) $$e^2 = 1 + \frac{1}{1!}\cdot 2 + \frac{1}{2!}\cdot 2^2 + \frac{1}{3!}\cdot 2^3 + \frac{1}{4!}\cdot 2^4 + \cdots .$$

(2) $$\sqrt[3]{e} = 1 + \frac{1}{1!}\cdot\frac{1}{3} + \frac{1}{2!}\cdot\left(\frac{1}{3}\right)^2 + \frac{1}{3!}\cdot\left(\frac{1}{3}\right)^3 + \frac{1}{4!}\cdot\left(\frac{1}{3}\right)^4 + \frac{1}{5!}\cdot\left(\frac{1}{3}\right)^5 + \cdots .$$

(3) $$e^{-1} = 1 + \frac{1}{1!}\cdot\left(-1\right) + \frac{1}{2!}\cdot\left(-1\right)^2 + \frac{1}{3!}\cdot\left(-1\right)^3 + \frac{1}{4!}\cdot\left(-1\right)^4 + \cdots$$

$$\left(= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots\right).$$