Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XVI

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§16. Negative exponents.

- I am still shooting for defining

$$2^x, \qquad 3^x, \qquad 4^x, \qquad \cdots$$

as functions on x. As innocuous as that may sound $\big($mainly because of the simplicity of the notation$\big)$, the same is indeed a taxing job. As a matter of fact, it requires us to address the feasibility of raising *irrational* number exponents. Here is the road map: In the next set of notes $\big($"Review of Lectures – XVII"$\big)$, we will define

$$2^{\sqrt{2}}.$$

Do you know what this signifies? We have proved $\big($in "Review of Lectures – XIII"$\big)$ that $\sqrt{2}$ is an irrational number. So what we have covered, of raising rational number exponents, does not apply. We will persevere all the technicalities and will thoroughly make sense of this number $2^{\sqrt{2}}$. The general case: a^x, where a is a positive real number, and x is any real number, will be easily extrapolated from it. It prompts us to define the exponential functions $\big($in "Review of Lectures – XVIII"$\big)$.

But let's hold onto that thought for now, because there is one thing that was left out which we need to take care of before getting into that point. Namely, within rational number exponents, we are yet to define negative exponents. As a starter:

Definition $\big((-1)$-th power$\big)$.

Assume a is a real number. Assume a is not equal to 0: $a \neq 0$. Then we define a^{-1} as $\dfrac{1}{a}$:

$$\boxed{a^{-1} \ = \ \frac{1}{a}} \, .$$

- Incorporating the above definition with $\dfrac{b}{a} = \dfrac{1}{a} \cdot b,$ we agree

$$\boxed{\frac{b}{a} = a^{-1} b.}\ .$$

Example 1.

$$1^{-1} = \frac{1}{1} = 1.$$

$$2^{-1} = \frac{1}{2}.$$

$$3^{-1} = \frac{1}{3}.$$

$$4^{-1} = \frac{1}{4}.$$

$$5^{-1} = \frac{1}{5}.$$

Pronunciations:

1^{-1}	:	"one inverse."
2^{-1}	:	"two inverse."
3^{-1}	:	"three inverse."
4^{-1}	:	"four inverse."
5^{-1}	:	"five inverse."
$\vdots$		$\vdots$

Definition $\big((-2)$-th power$\big)$.

Assume a is a real number. Assume a is not equal to 0: $a \neq 0$. Then we define a^{-2} as $\dfrac{1}{a^2}$:

$$\boxed{a^{-2} = \frac{1}{a^2}}\ .$$

Example 2.

$$1^{-2} = \frac{1}{1^2} = 1.$$

$$2^{-2} = \frac{1}{2^2} = \frac{1}{4}.$$

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}.$$

$$4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$$

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25}.$$

Definition $\big((-3)$-th power$\big)$.

Assume a is a real number. Assume a is not equal to 0: $a \neq 0$. Then we define a^{-3} as $\frac{1}{a^3}$:

$$\boxed{a^{-3} = \frac{1}{a^3}}\,.$$

Example 3.

$$1^{-3} = \frac{1}{1^3} = 1.$$

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$$

$$3^{-3} = \frac{1}{3^3} = \frac{1}{27}.$$

$$4^{-3} = \frac{1}{4^3} = \frac{1}{64}.$$

$$5^{-3} = \frac{1}{5^3} = \frac{1}{125}.$$

Definition $\big((-4)$**-th power**$\big)$**.**

Assume a is a real number. Assume a is not equal to 0: $a \neq 0$. Then we define a^{-4} as $\dfrac{1}{a^4}$:

$$\boxed{a^{-4} \;=\; \frac{1}{a^4}}\,.$$

Example 4.

$$1^{-4} \;=\; \frac{1}{1^4} \;=\; 1.$$

$$2^{-4} \;=\; \frac{1}{2^4} \;=\; \frac{1}{16}.$$

$$3^{-4} \;=\; \frac{1}{3^4} \;=\; \frac{1}{81}.$$

$$4^{-4} \;=\; \frac{1}{4^4} \;=\; \frac{1}{256}.$$

$$5^{-4} \;=\; \frac{1}{5^4} \;=\; \frac{1}{625}.$$

⋆ More generally:

Definition $\big((-n)$**-th power**$\big)$**.**

Assume a is a real number. Assume a is not equal to 0: $a \neq 0$. Assume n is a positive integer. Then we define a^{-n} as $\dfrac{1}{a^n}$:

$$\boxed{a^{-n} \;=\; \frac{1}{a^n}}\,.$$

Lemma 1. Assume a is a real number. Assume a is not equal to 0: $a \neq 0$. Then

$$\begin{aligned}
\left(\frac{1}{a}\right)^2 &= \frac{1}{a^2}, \\
\left(\frac{1}{a}\right)^3 &= \frac{1}{a^3}, \\
\left(\frac{1}{a}\right)^4 &= \frac{1}{a^4}, \\
\left(\frac{1}{a}\right)^5 &= \frac{1}{a^5}, \\
&\vdots
\end{aligned}$$

Proof.

$$\left(\frac{1}{a}\right)^2 = \frac{1}{a} \cdot \frac{1}{a} = \frac{1}{a \cdot a} = \frac{1}{a^2},$$

$$\begin{aligned}
\left(\frac{1}{a}\right)^3 = \left(\frac{1}{a}\right)^2 \cdot \frac{1}{a} &= \frac{1}{a^2} \cdot \frac{1}{a} \qquad \Big(\text{by the previous step}\Big) \\
&= \frac{1}{a^2 \cdot a} = \frac{1}{a^3},
\end{aligned}$$

$$\begin{aligned}
\left(\frac{1}{a}\right)^4 = \left(\frac{1}{a}\right)^3 \cdot \frac{1}{a} &= \frac{1}{a^3} \cdot \frac{1}{a} \qquad \Big(\text{by the previous step}\Big) \\
&= \frac{1}{a^3 \cdot a} = \frac{1}{a^4},
\end{aligned}$$

$$\begin{aligned}
\left(\frac{1}{a}\right)^5 = \left(\frac{1}{a}\right)^4 \cdot \frac{1}{a} &= \frac{1}{a^4} \cdot \frac{1}{a} \qquad \Big(\text{by the previous step}\Big) \\
&= \frac{1}{a^4 \cdot a} = \frac{1}{a^5},
\end{aligned}$$

and so on. □

Lemma 1 paraphrased. Assume a is a real number. Assume $a \neq 0$. Then

$$\begin{aligned}
\left(a^{-1}\right)^2 &= a^{-2}, \\
\left(a^{-1}\right)^3 &= a^{-3}, \\
\left(a^{-1}\right)^4 &= a^{-4}, \\
\left(a^{-1}\right)^5 &= a^{-5}, \\
&\vdots
\end{aligned}$$

Lemma 1 further paraphrased.

Assume a is a real number. Assume a is not equal to 0: $a \neq 0$. Assume n is a positive integer. Then

(1) $$\boxed{\left(\frac{1}{a}\right)^n = \frac{1}{a^n}}.$$

(2) $$\boxed{\left(a^{-1}\right)^n = a^{-n}}.$$

Lemma 2. Assume a and b are real numbers. Assume $b \neq 0$. Then

$$\begin{aligned}
\left(\frac{b}{a}\right)^2 &= \frac{b^2}{a^2}, \\
\left(\frac{b}{a}\right)^3 &= \frac{b^3}{a^3}, \\
\left(\frac{b}{a}\right)^4 &= \frac{b^4}{a^4}, \\
\left(\frac{b}{a}\right)^5 &= \frac{b^5}{a^5}, \\
&\vdots
\end{aligned}$$

(Proof of Lemma 2 is entirely parallel to that of Lemma 1, so I will skip it.)

Lemma 2 paraphrased. Assume a and b are real numbers. Assume $b \neq 0$. Then

$$\begin{aligned} \left(a^{-1}\,b\right)^2 &= a^{-2}\,b^2, \\ \left(a^{-1}\,b\right)^3 &= a^{-3}\,b^3, \\ \left(a^{-1}\,b\right)^4 &= a^{-4}\,b^4, \\ \left(a^{-1}\,b\right)^5 &= a^{-5}\,b^5, \\ &\vdots \end{aligned}$$

Lemma 2 further paraphrased. Assume a is a real number. Assume $a \neq 0$. Assume n is a positive integer. Then

(1) $$\boxed{\left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}}\,.$$

(2) $$\boxed{\left(a^{-1}\,b\right)^n = a^{-n}\,b^n}\,.$$

Lemma 3. Assume a and b are real numbers. Assume $a \neq 0$, $b \neq 0$. Then

$$\begin{aligned} \left(\frac{b}{a}\right)^{-1} &= \frac{a}{b}, \\ \left(\frac{b}{a}\right)^{-2} &= \frac{a^2}{b^2}, \\ \left(\frac{b}{a}\right)^{-3} &= \frac{a^3}{b^3}, \\ \left(\frac{b}{a}\right)^{-4} &= \frac{a^4}{b^4}, \\ &\vdots \end{aligned}$$

Lemma 3 paraphrased. Assume a and b are real numbers. Assume $a \neq 0$, $b \neq 0$. Then

$$\begin{aligned}
\left(a^{-1}\,b\right)^{-1} &= a\;b^{-1},\\
\left(a^{-1}\,b\right)^{-2} &= a^{2}\;b^{-2},\\
\left(a^{-1}\,b\right)^{-3} &= a^{3}\;b^{-3},\\
\left(a^{-1}\,b\right)^{-4} &= a^{4}\;b^{-4},\\
\vdots \quad & \quad \vdots
\end{aligned}$$

Lemma 3 further paraphrased. Assume a and b are real numbers. Assume $a \neq 0$, $b \neq 0$. Assume n is a positive integer. Then

(1)
$$\boxed{\left(\frac{b}{a}\right)^{-n} = \frac{a^n}{b^n}}\,.$$

(2)
$$\boxed{\left(a^{-1}\,b\right)^{-n} = a^n\;b^{-n}}\,.$$

- **Variations (of Lemma 3).**

$$\left(\frac{b}{a\;c}\right)^{n} = \frac{b^n}{a^n\;c^n}, \qquad \left(\frac{b\;d}{a\;c}\right)^{n} = \frac{b^n\;d^n}{a^n\;c^n},$$

$$\left(\frac{b\;c}{a}\right)^{-n} = \frac{a^n}{b^n\;c^n}, \qquad \left(\frac{b^3\;d}{a\;c^2}\right)^{-n} = \frac{a^n\;c^{2n}}{b^{3n}\;d^n},$$

$$\left(a\;b^{-1}\;c\right)^{n} = a^n\;b^{-n}\;c^n, \qquad \left(a^{-1}\;b^2\;c^{-1}\right)^{n} = a^{-n}\;b^{2n}\;c^{-n},$$

$$\left(a\;b^{-1}\;c\;d^{-1}\right)^{-n} = a^{-n}\;b^n\;c^{-n}\;d^n, \qquad \left(a^{-2}\;b^3\;c^{-4}\right)^{-n} = a^{2n}\;b^{-3n}\;c^{4n},$$

$$\vdots \qquad\qquad \vdots$$

- **Fractional negative exponents.**

At this point, nothing stops us from defining a^{-r} where r is a positive rational number. The following definition is exactly in the same outlook:

Definition $\Big((-r)$-th power, where r is a positive rational number$\Big)$.

Assume a is a real number. Assume $a \neq 0$. Assume r is a positive rational number. Then we define a^{-r} as $\dfrac{1}{a^r}$:

$$\boxed{a^{-r} \;=\; \frac{1}{a^r}}\,.$$

$\star$ Let n and k be two positive integers. Substitute $a = b^{\frac{1}{k}}$ into the identity $a^{-n} = \Big(a^n\Big)^{-1}$ (the definition of the $(-n)$-th power):

$$\left(b^{\frac{1}{k}}\right)^{-n} \;=\; \left(\left(b^{\frac{1}{k}}\right)^{n}\right)^{-1}.$$

Since the right-hand side of this last identity is simplifed as

$$\left(b^{\frac{n}{k}}\right)^{-1},$$

so

$$\left(b^{\frac{1}{k}}\right)^{-n} \;=\; \left(b^{\frac{n}{k}}\right)^{-1}.$$

The right-hand side of this is further rewritten as

$$\frac{1}{b^{\frac{n}{k}}}\,.$$

This is nothing but the definition of

$$b^{-\frac{n}{k}}.$$

Thus we obtained

$$\left(b^{\frac{1}{k}}\right)^{-n} = b^{-\frac{n}{k}} .$$

Now, if you substitute b with c^2, then

$$\left(\left(c^2\right)^{\frac{1}{k}}\right)^{-n} = \left(c^2\right)^{-\frac{n}{k}}.$$

This is rewritten as

$$\left(c^{\frac{2}{k}}\right)^{-n} = c^{-\frac{2n}{k}} .$$

More generally, let k, n and ℓ be positive integers, and

$$\left(c^{\frac{\ell}{k}}\right)^{-n} = c^{-\frac{\ell n}{k}} .$$

What this entails is that a part of the exponential law extends to the negative exponents, namely, the following holds when r is a positive rational number and s is a negative integer:

Rule III. $$\boxed{\left(c^r\right)^s = c^{rs}} .$$

In the above, r does not have to be positive. Also s does not have to be an integer. Moreover, other parts of the exponential law also extend to the negative exponent case as well. In particular,

- 'Lemma 1 further paraphrased' (page 6),
- 'Lemma 2 further paraphrased' (page 7),
- 'Lemma 3 further paraphrased' (page 8), and
- 'Variations (of Lemma 3) (page 8)

are all valid when n is a rational number. So, to highlight:

Exponential Laws (renewed). Let r and s be rational numbers (not necessarily positive). Let a and b be positive real numbers (not necessarily rational numbers). Then

Rule I. $$\boxed{\left(ab\right)^r = a^r\, b^r}\,.$$

Rule II. $$\boxed{a^r\, a^s = a^{r+s}}\,.$$

Rule III. $$\boxed{\left(a^r\right)^s = a^{rs}}\,.$$

Rule IV. $$\boxed{a^0 = 1}\,, \qquad \boxed{1^r = 1}\,.$$

Example 5. $2^{-\frac{1}{2}}$ means $\dfrac{1}{2^{\frac{1}{2}}}$, or the same $\dfrac{1}{\sqrt{2}}$.

Example 6. $2^{-\frac{3}{2}}$ means $\dfrac{1}{2^{\frac{3}{2}}}$, or the same $\dfrac{1}{2\sqrt{2}}$.

Example 7. $3^{-\frac{7}{4}}$ means $\dfrac{1}{3^{\frac{7}{4}}}$.

Example 8. $\left(\dfrac{10}{7}\right)^{-\frac{1}{2}}$ means $\left(\dfrac{7}{10}\right)^{\frac{1}{2}}$, or the same $\dfrac{7^{\frac{1}{2}}}{10^{\frac{1}{2}}}$.

Example 9. $\left(\dfrac{4}{5}\right)^{-\frac{2}{3}}$ means $\left(\dfrac{5}{4}\right)^{\frac{2}{3}}$, or the same $\dfrac{5^{\frac{2}{3}}}{4^{\frac{2}{3}}}$.

• So, basically we have taken care of raising the power of either positive, or negative, rational number exponents. But then how about something like

$$2^{\sqrt{2}}?$$

What does this mean? You might say this should be

$$2^{1.4142135623730950488016887242096980785696718753769...},$$

in view of

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769... \; .$$

Right? But since the decimal expression continues endlessly, so what does this mean? Keep in mind that, when we raise the power of a rational number, we utilize the fact a rational number can be written as "an integer divided by another integer" form. But we suddenly lose that because $\sqrt{2}$ is actually irrational. What that means is that $\sqrt{2}$ cannot be written as "an integer divided by another integer" form. (See "Review of Lectures – XIII".) Without such expression, we seemingly do not have a way to proceed. So sounds like we are doomed. Is there anthing can we do?

Yes, try the following:

$$
\begin{aligned}
&2^{1.4}, \\
&2^{1.41}, \\
&2^{1.414}, \\
&2^{1.4142}, \\
&2^{1.41421}, \\
&2^{1.414213}, \\
&2^{1.4142135}, \\
&2^{1.41421356}, \\
&2^{1.414213562}, \\
&2^{1.4142135623}, \\
&2^{1.41421356237}, \\
&2^{1.414213562373}, \\
&2^{1.4142135623730}, \\
&2^{1.41421356237309}, \\
&2^{1.414213562373095}, \\
&2^{1.4142135623730950}, \\
&2^{1.41421356237309504}, \\
&\vdots
\end{aligned}
$$

Now, each of these makes sense, because the exponent carries only finitely many digits, so they are rational numbers. But of course, these are all slightly off the 'target' number, what we want to define as $2^{\sqrt{2}}$. Good news, because we have investigated that when the two exponents are close then 2-to-the those exponents are also close. So as you move down in the above list the values of the numbers occupying those lines are supposedly getting closer and closer. But then with all this how do we exactly carve the 'target' number? — To be continued.