Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – XIII

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Instructor:　　　Yasuyuki Kachi

Line #:　52920.

§13.　Square roots.

● Today we learn <u>square-root/square-rooting</u> . First, notation and pronunciation:

$$\sqrt{0} \quad : \quad \text{``square \; root \; of \; zero.''}$$

$$\sqrt{1} \quad : \quad \text{``square \; root \; of \; one.''}$$

$$\sqrt{2} \quad : \quad \text{``square \; root \; of \; two.''}$$

$$\sqrt{3} \quad : \quad \text{``square \; root \; of \; three.''}$$

$$\sqrt{4} \quad : \quad \text{``square \; root \; of \; four.''}$$

$$\sqrt{5} \quad : \quad \text{``square \; root \; of \; five.''}$$

$$\sqrt{6} \quad : \quad \text{``square \; root \; of \; six.''}$$

$$\sqrt{7} \quad : \quad \text{``square \; root \; of \; seven.''}$$

$$\sqrt{8} \quad : \quad \text{``square \; root \; of \; eight.''}$$

Explaining these is not too straightforward. We revisit "Review of Lectures – V". As a starter, below is what you saw:

x	0	1	2	3	4	5	6	7	8	9
x^2	0	1	4	9	16	25	36	49	64	81

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Now, these mean

$$
\begin{aligned}
0^2 &= 0, \\
1^2 &= 1, \\
2^2 &= 4, \\
3^2 &= 9, \\
4^2 &= 16, \\
5^2 &= 25, \\
6^2 &= 36, \\
7^2 &= 49, \\
8^2 &= 64, \quad \text{and} \\
9^2 &= 81.
\end{aligned}
$$

These translate into

$$
\begin{aligned}
\sqrt{0} &= 0, \\
\sqrt{1} &= 1, \\
\sqrt{4} &= 2, \\
\sqrt{9} &= 3, \\
\sqrt{16} &= 4, \\
\sqrt{25} &= 5, \\
\sqrt{36} &= 6, \\
\sqrt{49} &= 7, \\
\sqrt{64} &= 8, \quad \text{and} \\
\sqrt{81} &= 9.
\end{aligned}
$$

Exercise 1. $\sqrt{100} =?$ $\sqrt{144} =?$ $\sqrt{225} =?$ $\sqrt{324} =?$

Consult the table below, if necessary.

x	10	11	12	13	14	15	16	17	18	19
x^2	100	121	144	169	196	225	256	289	324	361

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$\boxed{\textbf{Answers}}$: $\sqrt{100} = 10.$ Indeed, $100 = 10^2.$

$\sqrt{144} = 12.$ Indeed, $144 = 12^2.$

$\sqrt{225} = 15.$ Indeed, $225 = 15^2.$

$\sqrt{324} = 18.$ Indeed, $324 = 18^2.$

So, in short:

$\underline{\underline{\textit{If } \ n \ \textit{is a non-negative integer, and if}}} \quad \boxed{a = n^2} \quad \underline{\text{then}} \quad \boxed{\sqrt{a} = n}$.

So, basically, the square-rooting is the "inverse operation" to the squaring operation. Namely, suppose you square a number. Then square-root that outcome number. The effect is you "undo" the original squaring. What I mean is:

you square 2 $\implies$ answer is 4 $\implies$ you square-root 4 $\implies$ answer is 2.

you square 3 $\implies$ answer is 9 $\implies$ you square-root 9 $\implies$ answer is 3.

you square 4 $\implies$ answer is 16 $\implies$ you square-root 16 $\implies$ answer is 4.

you square 5 $\implies$ answer is 25 $\implies$ you square-root 25 $\implies$ answer is 5.

$\vdots$ $\quad\quad\quad$ $\vdots$ $\quad\quad\quad$ $\vdots$ $\quad\quad\quad$ $\vdots$

But the real issue here is,

$\sqrt{2} =?$ $\sqrt{3} =?$ $\sqrt{5} =?$ $\sqrt{6} =?$ $\sqrt{7} =?$ $\sqrt{8} =?$

$\sqrt{10} =?$ $\sqrt{11} =?$ $\sqrt{12} =?$ $\sqrt{13} =?$ $\sqrt{14} =?$ $\sqrt{15} =?$

$\sqrt{17} =?$ $\sqrt{18} =?$ $\sqrt{19} =?$ $\sqrt{20} =?$ $\sqrt{21} =?$ $\sqrt{22} =?$

$\sqrt{23} =?$ $\sqrt{24} =?$ $\sqrt{26} =?$ $\sqrt{27} =?$ $\sqrt{28} =?$ $\sqrt{29} =?$

etc. $\left(\text{as you can see, I excluded } \sqrt{0}, \ \sqrt{1}, \ \sqrt{4}, \ \sqrt{9}, \ \sqrt{16} \text{ and } \sqrt{25}\right).$

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- $\sqrt{2}$.

For example, $\sqrt{2}$ means a number whose square equals 2. Namely:

$$\text{``}\ \boxed{x = \sqrt{2}}\ \underline{\textit{is a number satisfying}}\ \boxed{x^2 = 2}\ \text{''}.$$

But do we know such a number? Does such a number exist? Today's discussion focuses on whether such a number really exists. The answer is, yes, such a number really exists. How do we find it? It goes step by step.

0. Observe

$$1^2 \ = \ 1, \qquad \longleftarrow \ \text{smaller than } 2$$
$$2^2 \ = \ 4. \qquad \longleftarrow \ \text{bigger than } 2$$

So $\sqrt{2}$ must sit between 1 and 2:

$$1 \ < \ \sqrt{2} \ < \ 2.$$

1. Observe

$$1.1^2 \ = \ 1.21,$$
$$1.2^2 \ = \ 1.44,$$
$$1.3^2 \ = \ 1.69,$$
$$1.4^2 \ = \ 1.96, \qquad \longleftarrow \ \text{smaller than } 2$$
$$1.5^2 \ = \ 2.25, \qquad \longleftarrow \ \text{bigger than } 2$$

So $\sqrt{2}$ must sit between 1.4 and 1.5:

$$1.4 \ < \ \sqrt{2} \ < \ 1.5.$$

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2. Observe

$$1.41^2 = 1.9881, \quad \longleftarrow \text{ smaller than } 2$$
$$1.42^2 = 2.0164, \quad \longleftarrow \text{ bigger than } 2$$

So $\sqrt{2}$ must sit between 1.41 and 1.42:

$$1.41 < \sqrt{2} < 1.42.$$

3. Observe

$$1.411^2 = 1.990921,$$
$$1.412^2 = 1.993744,$$
$$1.413^2 = 1.996569,$$
$$1.414^2 = 1.999396, \quad \longleftarrow \text{ smaller than } 2$$
$$1.415^2 = 2.002225. \quad \longleftarrow \text{ bigger than } 2$$

So $\sqrt{2}$ must sit between 1.414 and 1.415:

$$1.414 < \sqrt{2} < 1.415.$$

4. Observe

$$1.4141^2 = 1.99967881,$$
$$1.4142^2 = 1.99996164, \quad \longleftarrow \text{ smaller than } 2$$
$$1.4143^2 = 2.00024449. \quad \longleftarrow \text{ bigger than } 2$$

So $\sqrt{2}$ must sit between 1.4142 and 1.4143:

$$1.4142 < \sqrt{2} < 1.4143.$$

5. Observe

$$1.41421^2 \;=\; 1.9999899241, \qquad \longleftarrow \;\; \text{smaller than } 2$$

$$1.41422^2 \;=\; 2.0000182084. \qquad \longleftarrow \;\; \text{bigger than } 2$$

So $\sqrt{2}$ must sit between 1.41421 and 1.41422:

$$1.41421 \;<\; \sqrt{2} \;<\; 1.41422.$$

6. Observe

$$1.414211^2 \;=\; 1.999992752521,$$

$$1.414212^2 \;=\; 1.999995580944,$$

$$1.414213^2 \;=\; 1.999998409369, \qquad \longleftarrow \;\; \text{smaller than } 2$$

$$1.414214^2 \;=\; 2.000001237796. \qquad \longleftarrow \;\; \text{bigger than } 2$$

So $\sqrt{2}$ must sit between 1.414213 and 1.414214:

$$1.414213 \;<\; \sqrt{2} \;<\; 1.414214.$$

So,

$$\sqrt{2} \;=\; 1.414213... \;\; .$$

But of course, the above is only up to the sixth digit under decimal point of $\sqrt{2}$. In other words, $\sqrt{2}$ does not exactly equal 1.414213. But you can determine the seventh, eighth, nineth, ... digits, by simply continuing the above procedure. Indeed, the above procedure continues endlessly. If you want to see more digits:

$$\sqrt{2} \;=\; 1.41421356237309504880168872420969807856696718753769... \;\; .$$

Most importantly, the decimal expression of $\sqrt{2}$ continues forever, it never ends.

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Once again, the above algorithm theoretically determines the digit of the decimal expression of $\sqrt{2}$ at any place under the decimal point.

But don't you want to know a more efficient algorithm? Here, please don't say "hey, how about calculator". There is a method to calculate the square root of a decimally expressed number, and the output is also written in decimals. Below is the calulation of $\sqrt{2}$ using that method. When it comes to square-rooting, your calculator basically uses the same algorithm (or a minor variation of it). So this is worth taking a look at. What's underlying this method is — not too surprisingly — Binomial Formula, though it does not show up explicitly. So, Binomial Formula is indeed our recurring theme.

$$
\begin{array}{c}
\boxed{1}\,.\ \boxed{4}\quad \boxed{1}\quad \boxed{4}\quad \boxed{2}\ ... \\
\sqrt{\ \ 2\,.\ 0\ \ 0\quad 0\ \ 0\quad 0\ \ 0\quad 0\ \ 0}
\end{array}
$$

$\boxed{1}$	$\longleftarrow$ (line 0)
$\boxed{1}\ \boxed{0}\ \boxed{0}$ $\boxed{9}\ \boxed{6}$	$\longleftarrow$ (line 1)
$\boxed{4}\ \boxed{0}\ \boxed{0}$ $\boxed{2}\ \boxed{8}\ \boxed{1}$	$\longleftarrow$ (line 2)
$\boxed{1}\ \boxed{1}\ \boxed{9}\ \boxed{0}\ \boxed{0}$ $\boxed{1}\ \boxed{1}\ \boxed{2}\ \boxed{9}\ \boxed{6}$	$\longleftarrow$ (line 3)
$\boxed{6}\ \boxed{0}\ \boxed{4}\ \boxed{0}\ \boxed{0}$ $\boxed{5}\ \boxed{6}\ \boxed{5}\ \boxed{6}\ \boxed{4}$	$\longleftarrow$ (line 4)
$\boxed{3}\ \boxed{8}\ \boxed{3}\ \boxed{6}$	

$\vdots$ $\ddots$

Let's dissect.

○   Start with

$$\sqrt{\boxed{a}\,.\,\boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \cdots}$$

$$\sqrt{2\,.\,0\ 0\quad 0\ 0\quad 0\ 0\quad 0\ 0}$$

$\boxed{}$ $\longleftarrow$ (line 0)

○   You see a on top. a is one of 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. We are going to decide a.

○   Choose the largest a such that a^2 does not exceed 2. So $a = 1$. Register your answer $a = 1$ on top. At the same time, place $a^2 = 1$ in (line 0) as indicated. Subtract (line 0) from the line right above it:

$$\sqrt{\boxed{1}\,.\,\boxed{b}\ \boxed{}\ \boxed{}\ \boxed{}\ \cdots}$$

$$\sqrt{2\,.\,0\ 0\quad 0\ 0\quad 0\ 0\quad 0\ 0}$$

$\boxed{1}$

$\boxed{1}\ \boxed{0}\ \boxed{0}$

$\boxed{}\ \boxed{}\ \boxed{}$ $\longleftarrow$ (line 1)

○   Now the subtraction was performed. 00 was dragged down from the top. At this point you see 100 right above (line 1).

○   Now you see b on top. b is one of 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. We are going to decide b.

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○   Choose the largest b such that

$$20 \cdot \boxed{1} \cdot b + b^2$$

does not exceed 100, where $\boxed{1}$ is in the left of $\boxed{b}$. So $b = 4$. Register your answer $b = 4$ on top. At the same time, place $20 \cdot 1 \cdot b + b^2 = 96$ in (line 1) as indicated. Subtract (line 1) from the line right above it:

○   Now the subtraction was performed. 00 was dragged down from the top. At this point you see 400 right above (line 2).

○   Now you see c on top. c is one of 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. We are going to decide c.

○   Choose the largest c such that

$$20 \cdot \boxed{14} \cdot c + c^2$$

does not exceed 400, where $\boxed{14}$ is in the left of $\boxed{c}$. So $c = 1$. Register your answer $c = 1$ on top. At the same time, place $20 \cdot 14 \cdot c + c^2 = 281$ in (line 2) as indicated. Subtract (line 2) from the line right above it:

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$$\begin{array}{r} \boxed{1}\,.\quad \boxed{4}\quad \boxed{1}\quad \boxed{d}\quad \boxed{}\,\ldots \\ \sqrt{2\,.\,0\ 0\quad 0\ 0\quad 0\ 0\quad 0\ 0} \end{array}$$

$$\boxed{1}$$

$$\boxed{1}\quad \boxed{0}\ \boxed{0}$$
$$\boxed{9}\ \boxed{6}$$

$$\boxed{4}\quad \boxed{0}\ \boxed{0}$$
$$\boxed{2}\quad \boxed{8}\ \boxed{1}$$

$$\boxed{1}\ \boxed{1}\ \boxed{9}\ \boxed{0}\ \boxed{0}$$
$$\boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}$$

$\longleftarrow$ (line 3)

○ Now the subtraction was performed. 00 was dragged down from the top. At this point you see 11900 right above (line 3).

○ Now you see d on top. d is one of 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. We are going to decide d.

○ Choose the largest d such that

$$20 \cdot \boxed{141} \cdot d \quad + \quad d^2$$

does not exceed 11900, where $\boxed{141}$ is in the left of $\boxed{d}$. So $d = 4$. Register your answer $d = 4$ on top. At the same time, place $20 \cdot 141 \cdot d + d^2$ $=$ 11296 in (line 3) as indicated. Subtract (line 3) from the line right above it:

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$$1 . \quad 4 \quad 1 \quad 4 \quad \square \ \ldots$$

$$\sqrt{} \quad 2 . 0 \ 0 \quad 0 \ 0 \quad 0 \ 0 \quad 0 \ 0$$

$$1$$

$$1 \quad 0 \ 0$$
$$9 \ 6$$

$$4 \quad 0 \ 0$$
$$2 \quad 8 \ 1$$

$$1 \quad 1 \quad 9 \quad 0 \ 0$$
$$1 \quad 1 \quad 2 \quad 9 \ 6$$

$$6 \quad 0 \ 4 \quad 0 \ 0$$
$$\square \quad \square \ \square \quad \square \ \square$$

$$\longleftarrow \quad \text{(line 4)}$$

And so on so forth.

You can continue this procedure, and get as many digits under the decimal point as you want for the number $\sqrt{2}$. The computation becomes harder as you move on, though. Indeed, in this method, the size of the number you have to deal with (in terms of how many digits it carries) grows proportionately to the number of all the past steps.

Exercise 2. Use the same algorithm to find the decimal expression of each of $\sqrt{3}$ and $\sqrt{5}$, up to the fourth place under the decimal point.

Exercise 3. Use the same algorithm to find the decimal expression of $\sqrt{e}$, up to the fourth place under the decimal point. Here,

$$e \ = \ 2.71828182\ldots \ .$$

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$$\boxed{1}\,.\quad \square \qquad \square \qquad \square \qquad \square \,\text{...}$$

$$\sqrt{}\ 3\,.\ 0\ \ 0\quad 0\ \ 0\quad 0\ \ 0\quad 0\ \ 0$$

$\boxed{1}$

(long division work with blank boxes)

$\vdots \qquad\qquad \ddots$

★   The answer is found in page 22.

$$\boxed{2}\ .\quad \square \qquad \square \qquad \square \qquad \square \ \text{...}$$

$$\sqrt{}\quad 5\ .\ 0\ \ 0 \quad 0\ \ 0 \quad 0\ \ 0 \quad 0\ \ 0$$

$\boxed{4}$

$\square$

⋮   ⋱

★   The answer is found in page 23.

$\boxed{\textbf{Worksheet for Exercise 3}}$: $\sqrt{e}$.

$$\boxed{1}\,.\quad \square\quad \square\quad \square\quad \square\ \text{...}$$

$$\sqrt{}\quad 2\,.\ 7\quad 1\quad 8\quad 2\quad 8\quad 1\quad 8\quad 2$$

$\boxed{1}$

⋮   ⋱

★   The answer is found in page 24.

- **$\sqrt{2}$ is an irrational number.**

Once upon a time, there was a vexing problem, to decide whether $\sqrt{2}$ is written as $\dfrac{k}{m}$ using two integers k and m. The answer is, no, no matter how you cleverly arrange a pair of integers k and m, you cannot make the ratio $\dfrac{k}{m}$ equal to $\sqrt{2}$.

Geometrically, $\sqrt{2}$ is the length of the diagonal of a square whose edges have the unit length. It is also by virtue of this fact that historically the number $\sqrt{2}$ has been under scrutiny. But let's not talk about geometry (yet).

The above question is actually very old. We are not talking about a few centuries old. Babylonian clay tablet (c. 1800–1600 B.C.) contained an approximation of $\sqrt{2}$ as

$$ 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \left(= \frac{30547}{21600} = 1.41421296296296... \right).* $$

A different approximation of $\sqrt{2}$ was discovered in ancient India, in the pre-Pythagorean era (more about the Pythagorean school below).* While the above is a finite expression that only uses the usual arithmetic, namely, addition, subtraction, multiplication and division, and only involving integers as ingredients, it is important to agree that this is an *approximation* of $\sqrt{2}$. Babylonians did not know if there would be an *exact*, finite, expression of $\sqrt{2}$ with the same feature, namely, one that only uses the usual arithmetic (addition, subtraction, multiplication and division), and only involving integers as ingredients. Since you can always reduce such a formation into a single fraction, where both the numerator and the denominator are integers

$$ \left(\text{for example, in the above} \quad 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \quad \text{was reduced to} \quad \frac{30547}{21600} \right), $$

the same question is paraphrased as follows: whether it is true that $\sqrt{2}$ is expressed as a fraction of the form an integer divided by another integer. Babylonians did not know the answer to that. I am not sure whether they were aware of the fact that this question makes sense, and that it needs to be answered.

Then once there was an academy called "Pythagorean school" in ancient Greece (c. 500 B.C.). They answered this question. They discovered that $\sqrt{2}$ *cannot* expressed as a fraction of the form an integer divided by another integer.

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*I pulled this info from Wikipedia.
*Also from Wikipedia.

In the modern language, we say

$$\text{``}\ \underline{\underline{\sqrt{2}\ \text{ is not a } \textit{rational} \text{ number.}}}\ \text{''}$$

Or we also say

$$\text{``}\ \underline{\underline{\sqrt{2}\ \text{ is an } \textit{irrational} \text{ number.}}}\ \text{''}$$

Here, the meaning of a number being <u>rational</u> and <u>irrational</u> is as follows:

Definition (Rational numbers).

<u>A rational number</u> is a number which can be written as

$$\frac{k}{m} \qquad \left(k,\ m\text{: an integer,} \quad m \neq 0 \right).$$

★ Note that an integer is a rational number, because an integer k can be written as $\dfrac{k}{1}$.

Example 1. $\quad \dfrac{1}{2}, \quad \dfrac{2}{3}, \quad 5, \quad \dfrac{7}{3}, \quad -2, \quad 0, \quad \dfrac{3}{10}, \quad -\dfrac{11}{6}, \quad -1000,$

are all examples of rational numbers.

Now, there is an alternative definition of a rational number:

Alternative Definition (Rational numbers).

<u>A rational number</u> is a number which falls into either (i) or (ii):

(i)   its decimal expression stops after finitely many digits under the decimal point (this includes an integer), or

(ii)   its decimal expression contains a portion ('unit') made of a finite number of consecutive digits, and the whole decimal expression of that number is an <u>infinite times iteration</u> of that unit, except possibly a finite number of leading digits.

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Example 2.

(1) $0.52 = \dfrac{13}{25}$,   and $0.3125 = \dfrac{5}{16}$

are both rational numbers. These fall into (i).

(2) $0.003636363636.... = \dfrac{1}{275}$,   and $10.9\underline{142857}\,\underline{142857}\,\underline{142857}... = \dfrac{764}{70}$

are both rational numbers. These fall into (ii).

Definition (Irrational numbers).

A number which is *not* a rational number is called   an irrational number .

Example 3. $\sqrt{2}$ is an irrational number.

Example 4. $\sqrt{3}$ is an irrational number.

Example 5.   More generally, $\sqrt{n}$ where n is a positive integer and is not written
as $n = a^2$ using another integer a, is an irrational number.

So, in particular, all of

$$\sqrt{2}, \qquad \sqrt{3}, \qquad \sqrt{5}, \qquad \sqrt{6}, \qquad \sqrt{7}, \qquad \sqrt{8},$$

$$\sqrt{10}, \qquad \sqrt{11}, \qquad \sqrt{12}, \qquad \sqrt{13}, \qquad \sqrt{14}, \qquad \sqrt{15},$$

$$\sqrt{17}, \qquad \sqrt{18}, \qquad \sqrt{19}, \qquad \sqrt{20}, \qquad \sqrt{21}, \qquad \sqrt{22},$$

$$\sqrt{23}, \qquad \sqrt{24}, \qquad \sqrt{26}, \qquad \sqrt{27}, \qquad \sqrt{28}, \qquad \sqrt{29},$$

are irrational numbers.

- Next, I am going to show you how to logically refute the possibility $\sqrt{2}$ is a rational number. Let me use some technical term mathematicians use:

$$\text{prove } (v), \qquad \text{proof } (n).$$

The sentence below is how we usually say it:

"We are going to *prove* the fact that $\sqrt{2}$ is an *irrational* number."

Or sometimes instead

"We are going to *give a proof* of the fact that $\sqrt{2}$ is a *irrational* number."

Or sometimes instead

"We are going to *disprove* the possibility $\sqrt{2}$ is a *rational* number."

The commonly accepted style is you need to first state the statement, and then give a proof. The statement part is called

"theorem."

Then what comes after stating 'theorem' is, of course,

"proof."

Your proof has to be logically consistent, flawless and irrefutable. 99 percent consistent, flawless and irrefutable is insufficient. It has to be absolutely 100 percent logically consistent, flawless and irrefutable. The proof I give below is indeed 100 percent logically consistent, flawless and irrefutable.

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Theorem $\left(\text{Irrationality of } \sqrt{2}\,\right).$

$$\boxed{\sqrt{2} \text{ is irrational.}}$$

$\star$ The method of proof I employ here is called

$$\text{``proof by contradiction.''}$$

Proof.   Suppose $\sqrt{2}$ is written as

$$\sqrt{2} \;=\; \frac{k}{m}$$

using some integers k and m $\left(\text{where } m \neq 0\right).$

First, if both k and m are even, then we may simultaneously divide both the numerator and the denominator by 2 $\left(\text{and the value of the fraction stays the same}\right).$ After that procedure, suppose both the numerator and the denominator still remain to be even, then we repeat the same procedure as many times as necessary until at least one of the numerator and the denominator becomes odd. Thus we may assume, without loss of generality, that at least one of k and m is odd.

Under this assumption, square the both sides of the identity $\sqrt{2} \;=\; \dfrac{k}{m},$ thus

$$2 \;=\; \frac{k^2}{m^2}.$$

This is the same as

$$2\,m^2 \;=\; k^2.$$

The left-hand side of this last identity is clearly even, so this last identity forces its right-hand side to be also even. That in turn implies k is even, because if k is odd then k^2 is odd. But then k being even implies k^2 is divisible by 4. So by virtue of the above last identity $2\,m^2$ is divisible by 4, or the same to say, m^2 is divisible by 2, or the same to say, m^2 is even. This implies that m is even. In short, both k and m are even. This contradicts our assumption, that at least one of k and m is odd. The proof is complete. $\square$

How was that? You have just seen the glimpse of how mathematical proof works. Now, let me say something that is potentially confusing:

Quick tutorial. <u>Mathematics is ultimately all about proofs.</u>

What this means is as follows: Theorems and proofs almost entirely occupy any mathematical paper published in any scientific journal. This may be shocking to some. People believe that math is entirely calculations. I briefly touched this subject, that there is a myth that higher math is just manipulations of numbers (especially large numbers). Contrary to such reputations of math, mathematicians' main job is actually to devise proofs.

To give you some nuts-and-bolts, here is how it works: You are writing up a mathematical research paper, and intend to submit it to a scientific journal for publication. In it, suppose you make a certain mathematical statement, and call it a theorem. Then you are obligated to do either one of the following: You either cite the exact article that originally gave a proof of that thereom, or, in case there is no existing article that contains a proof of the same theorem, you have to provide a proof of it yourself. Suppose the latter is the case, and suppose you actually provide a proof of it, then that's called a 'new theorem'. Now, a paper without new theorems, or without valid proofs of new theorems, is generally considered not acceptable for publication. However, there are exceptions, namely, some journals publish 'expositions', in which case an inclusion of new theorems is not a requirement. Now, there is another word for a mathematical statement, which has an entirely different meaning:

<div align="center">"conjecture."</div>

A conjecture is a mathematical statement proclaimed in a research article without proof, with a clear annotation 'conjecture', and no existing article contains a proof of that statement. The mathematicians' community's stance is, whether that statement is mathematically true or false is open, it is up in the air. "Riemann Hypothesis" is an example of a conjecture. Indeed, as of today (February 18, 2015) we don't know yet whether "Riemann Hypothesis" is true, or false. Now, as the time progresses somebody may devise a proof of a conjecture, publish an article and spell out the proof of it. Once that article is accepted for publication, then the status of that mathematical statement is changed from a 'conjecture' to a 'theorem'.

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What about computers? You like to hear it or not, computers do not have enough intelligence to provide proofs to most mathematical statements. (However, please see the passage about 'Four Color Problem', in the very first set of notes "Review of Lectures – I", middle paragraph of page 7.) Indeed, there is no set algorithm to generate a proof of each given mathematical statement. Or, there is no 'template' for proofs. So, devising a proof is always "trial and error". That's also the reason why a mathematician cannot project how much time it would take before the next paper be finalized, or it would ever be finalized. Here, this last part "or it would ever be finalized" is not just an innocuous remark, but I am making a reference to one specific "meta-theorem", widely known in the mathematicians' community as "Gödel's incompleteness theorem". What it says is basically that there are mathematical statements that are "indecidable", namely, it is theoretically impossible to neither prove nor disprove that statement. Moreover, another part of the same meta-theorem asserts that, there is no way to know in advance which mathematical statements fall into that "indecidable" category. By the way, that meta-theorem was indeed proved by Gödel,* so it is known to be correct. What that entails is that, in mathematical research, there is no guarantee that your time invested will always be rewarded. You try to prove some conjecture, so you invest years of your time, so literally it becomes your life-work, but the risk you are running is huge because of the possibility that that conjecture happens to be "indecidable". Your problem being "indecidable" means your investment of time and energy is completely wasted. What's so vexing is, neither you, Gödel, nor anyone else can tell if the problem you are tackling is "indecidable". So, for example, who knows, "Riemann Hypothesis" may actually be "indecidable". There is no way to know if that is the case, unless someone actually proves it. Now, all that said, statistically speaking, most mathematicians navigate their careers by way of somehow choosing "decidable" problems, as in they end up proving new theorems and publish papers out of them, which only retrospectively proves that those theorems in your paper (which *became* theorems thanks to your paper) were indeed "decidable". To a mathematician, a research paper written by herself/himself is her/his "brain child".

When I said mathematicians's job never becomes obsolete, or when I said we need at least quarter million mathematicians in the entire world, what I meant is exactly this. On the other hand, the above theorem, that $\sqrt{2}$ is irrational, is the most basic kind of a theorem, which every mathematician can recite the proof by heart. This is indeed the most elementary kind of proof you see in the entire undergraduate math curriculum.

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*Kurt Gödel (1906–1978).

$\boxed{\textbf{Answer for Exercise 2, (a)}}$:

$$\sqrt{3} \quad = \quad 1.7320...$$

$\boxed{\textbf{Work}}$:

$$\sqrt{3} = 1.7320...$$

1 . 7 3 2 0 ...

$\sqrt{3 \ . \ 0 \ 0 \quad 0 \ 0 \quad 0 \ 0 \quad 0 \ 0}$

1		
2	0	0
1	8	9

1	1	0	0
1	0	2	9

7	1	0	0
6	9	2	4

1	7	6	0	0
				0

| 1 | 7 | 6 | 0 | 0 |

$\vdots \qquad\qquad \ddots$

$$\sqrt{5} \;=\; 2.2360...$$

$\boxed{\textbf{Work}}$:

Answer for Exercise 3 :

$$\sqrt{e} \quad = \quad 1.6487...$$

Work :

$$\begin{array}{c}
\boxed{1}.\quad\boxed{6}\quad\quad\boxed{4}\quad\quad\boxed{8}\quad\quad\boxed{7}\,...\\
\sqrt{\quad 2\,.\,7\quad 1\quad 8\quad 2\quad 8\quad 1\quad 8\quad 2}
\end{array}$$

$$\boxed{1}$$

$$\boxed{1}\;\boxed{7}\;\boxed{1}$$
$$\boxed{1}\;\boxed{5}\;\boxed{6}$$

$$\boxed{1}\;\boxed{5}\quad\boxed{8}\;\boxed{2}$$
$$\boxed{1}\;\boxed{2}\quad\boxed{9}\;\boxed{6}$$

$$\boxed{2}\quad\boxed{8}\;\boxed{6}\quad\boxed{8}\;\boxed{1}$$
$$\boxed{2}\quad\boxed{6}\;\boxed{3}\quad\boxed{0}\;\boxed{4}$$

$$\boxed{2}\;\boxed{3}\quad\boxed{7}\;\boxed{7}\quad\boxed{8}\;\boxed{2}$$
$$\boxed{2}\;\boxed{3}\quad\boxed{0}\;\boxed{7}\quad\boxed{6}\;\boxed{9}$$

$$\boxed{7}\;\boxed{0}\quad\boxed{1}\;\boxed{3}$$

$$\vdots\qquad\qquad\ddots$$

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