Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – XII

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§12. Factorials. Definition of e.

• Do you remember we briefly touched $\big($in "Reivew of Lectures – VIII"$\big)$ the following:

$$\begin{aligned}
1 &= 1, \\
2 &= 1 \cdot 2, \\
6 &= 1 \cdot 2 \cdot 3, \\
24 &= 1 \cdot 2 \cdot 3 \cdot 4, \\
120 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5, \\
720 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6, \\
5040 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7, \\
40320 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8, \\
362880 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9, \\
3628800 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10, \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddots
\end{aligned}$$

This is not just another sequence of numbers. This particular sequence is extremely fundamental in mathematics, as in it shows up everywhere, it is impossible to avoid it. If you like, we can certainly write these in the reverse multiplication fashion:

$$
\begin{aligned}
1 &= 1, \\
2 &= 2 \cdot 1, \\
6 &= 3 \cdot 2 \cdot 1, \\
24 &= 4 \cdot 3 \cdot 2 \cdot 1, \\
120 &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \\
720 &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \\
5040 &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \\
40320 &= 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \\
362880 &= 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \\
3628800 &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \ddots
\end{aligned}
$$

These numbers appear so frequently in mathematics that they have a name:

"factorial numbers."

What's more, there is a convenient way to write these numbers in a short form, using the symbol

$!$: "the factorial symbol."

This is the same as the exclamation symbol. But in math, we never call it the exclamation symbol. Also, please do not inadvertently place the exclamation symbol after a number for emphasis, because it will be confused with the factorial. Now, we denote the above ten numbers as

$$1!, \quad 2!, \quad 3!, \quad 4!, \quad 5!, \quad 6!, \quad 7!, \quad 8!, \quad 9! \ \text{ and } \ 10!,$$

respectively. So

$$
\begin{aligned}
1! &= 1,\\
2! &= 2\cdot 1,\\
3! &= 3\cdot 2\cdot 1,\\
4! &= 4\cdot 3\cdot 2\cdot 1,\\
5! &= 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
6! &= 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
7! &= 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
8! &= 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
9! &= 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
10! &= 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\\
&\vdots \qquad\qquad\qquad\qquad\qquad \ddots
\end{aligned}
$$

Pronunciation:

$$
\begin{aligned}
1\,! &= \text{“one factorial”},\\
2\,! &= \text{“two factorial”},\\
3\,! &= \text{“three factorial”},\\
4\,! &= \text{“four factorial”},\\
5\,! &= \text{“five factorial”},\\
6\,! &= \text{“six factorial”},\\
7\,! &= \text{“seven factorial”},\\
8\,! &= \text{“eight factorial”},\\
9\,! &= \text{“nine factorial”}.\\
10\,! &= \text{“ten factorial”}.
\end{aligned}
$$

So, can you tell me how to pronounce $20!$? Yes, "twenty factorial". Can you spell that out, as in can you describe that number without '!' ? Yes,

$$\begin{aligned} 20! \quad &= \quad 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \\ &\quad \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \,. \end{aligned}$$

Do you want to know how much this is? This is

$$\begin{aligned} 20! \quad &= \quad 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \\ &\quad \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= \quad 2432902008176640000. \end{aligned}$$

This one I did by hand.

So, as you can imagine, the sequence of factorial numbers grow very rapidly. To use a real life example:

Metaphor. You started a lemonade business. Words spread fast and the sales dramatically increased (Table 1 below):

day	$
1	1
2	2
3	6
4	24
5	120
6	720
⋮	⋮

On Day n, your sales will be n times the previous day sales. Suppose the same patterns hold until Day 50. Then the sales on Day 50 is how much?

Yes, the dollar amount of the sales on Day n is $n\,!$ $\big($"n factorial"$\big)$. So the dollar amount of the sales on Day 50 is exactly $50!$ $\big($"fifty factorial"$\big)$:

$$\begin{aligned} 50\,! \quad = \quad & 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43 \cdot 42 \cdot 41 \\ & \cdot 40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31 \\ & \cdot 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \\ & \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \\ & \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1\,. \end{aligned}$$

Hey, but if you are the owner of this business, you want to know the ball-park figure of how much this is, for asset estimating and portfolio management, possibly for your future stock investment. For that, we can certainly rely on our computer. Here is what my computer software $\big($Maple$\big)$ spit out:

$$\begin{aligned} 50\,! \quad = \quad & 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43 \cdot 42 \cdot 41 \\ & \cdot 40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31 \\ & \cdot 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \\ & \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \\ & \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

$$= 30414093201713378043612608166064768844377641568960512000000000000.$$

This is a 64-digit number. Wow. But of course, this is a 'dreamlike' scenario, it is surreal, it does not mirror the reality. So understand the above as a mathematical model. As a mathematical model, though, it makes perfect sense. It is just one of a handful down-to-earth ways off the top of my head to make your impression of the concept of factorials more evocative. The main thrust of the concept of fatorials is genuinely mathematical. So, let's put the metaphor part $\big($the part where the numbers represent money$\big)$ aside, and let's talk about the mathematical side of it.

So, $50!$. First, I think my computer did this one brute-force, as in it just multiplied 1 through 50. It did it in a split-second. Not so surprising. The truth is, it will be different for the factorials of larger numbers, like $1000000!$, or $1000000000!$.

First of all, the numbers of digits for numbers like $1000000!$, and $1000000000!$, are astronomical, so it is not possible to display on the screen the entire answer. So your computer would provide the ball-park figure. If it were to do those brute-force, it would take forever to finish the job. But if you enter something like $1000000000!$, your computer can still perform it and gives the answer $\big($which is, like I said, a ball-park figure$\big)$ in a matter of split-seconds. How do they do that? Do you want to know?

Sure. The next thing I say is very important, so listen: The moment it recognizes the problem, it immediately says to itself "forget brute-force calculation", to multiply 1 through one million. "No way, that's beyond my capacity." But then it quickly identifies the theorem pre-installed that helps computing the figure with the least amount of calculations. And that theorem itself is actually old, was devised around 1730. So, there is actually some "theory behind computing factorials". In order to reveal what that is, somehow, we need to study more closely about the behavior of

$$\left(1+\frac{1}{n}\right)^n; \qquad n = 1, 2, 3, 4, 5, \cdots$$

as n grows larger. We already know two things about this sequence. (a) This is an increasing sequence. (b) No matter how large n is, the number cannot exceed 3. You maight feel I am rather abruptly changing the subject, this is seemingly unrelated to the factorials, but not so fast. The following simple observation is actually very helpful toward our goal:

" *When you write a binomial coefficient in a fraction form, the denominator is a factorial number.* "

Example.

$$\binom{7}{3} = \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3} = \frac{7\cdot 6\cdot 5}{3!},$$

$$\binom{9}{5} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{5!},$$

$$\binom{10}{4} = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} = \frac{10\cdot 9\cdot 8\cdot 7}{4!}.$$

More generally:

- **Binomial coefficients expressed in terms of factorials — I.**

Let n and k be integers, with $0 < k < n$. Then the binomial coefficient $\binom{n}{k}$ is written as

$$\boxed{\binom{n}{k} \quad = \quad \frac{n\ \big(n-1\big)\ \big(n-2\big)\ \cdots\ \big(n-k+1\big)}{k\,!}} \, .$$

Now, from this perspective let's revisit "Review of Lectures – X". We binomially expanded $\left(1+\frac{1}{5}\right)^5$, as in

$$(5) \qquad \left(1+\frac{1}{5}\right)^5 \quad = \quad \binom{5}{0} \cdot 1^5$$
$$+ \binom{5}{1} \cdot 1^4 \cdot \left(\frac{1}{5}\right)$$
$$+ \binom{5}{2} \cdot 1^3 \cdot \left(\frac{1}{5}\right)^2$$
$$+ \binom{5}{3} \cdot 1^2 \cdot \left(\frac{1}{5}\right)^3$$
$$+ \binom{5}{4} \cdot 1 \cdot \left(\frac{1}{5}\right)^4$$
$$+ \binom{5}{5} \cdot \left(\frac{1}{5}\right)^5 \quad .$$

We rewrote this as

$$
\begin{aligned}
&1\\
&+ \frac{5}{1} \cdot \frac{1}{5}\\
&+ \frac{5 \cdot 4}{1 \cdot 2} \cdot \frac{1}{5 \cdot 5}\\
&+ \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} \cdot \frac{1}{5 \cdot 5 \cdot 5}\\
&+ \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{1}{5 \cdot 5 \cdot 5 \cdot 5}\\
&+ \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{1}{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5},
\end{aligned}
$$

and further rewrote this as

$$
\begin{aligned}
&1\\
&+ \frac{1}{1} \cdot \frac{5}{5}\\
&+ \frac{1}{1 \cdot 2} \cdot \frac{5}{5} \cdot \frac{4}{5}\\
&+ \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5}\\
&+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5}\\
&+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5}.
\end{aligned}
$$

Incorporating the factorial notation, this is

$$
\begin{aligned}
& 1 \\
&+ \frac{1}{1!} \cdot \frac{5}{5} \\
&+ \frac{1}{2!} \cdot \frac{5}{5} \cdot \frac{4}{5} \\
&+ \frac{1}{3!} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \\
&+ \frac{1}{4!} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \\
&+ \frac{1}{5!} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5}\,.
\end{aligned}
$$

So, in short,

$$
\left(1+\frac{1}{5}\right)^5 \tag{5}
$$

$$
\begin{aligned}
= \quad & 1 && \text{(term 5-0)} \\
&+ \frac{1}{1!} \cdot \frac{5}{5} && \text{(term 5-1)} \\
&+ \frac{1}{2!} \cdot \frac{5}{5} \cdot \frac{4}{5} && \text{(term 5-2)} \\
&+ \frac{1}{3!} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} && \text{(term 5-3)} \\
&+ \frac{1}{4!} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} && \text{(term 5-4)} \\
&+ \frac{1}{5!} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5}\,. && \text{(term 5-5)}
\end{aligned}
$$

What's clear is

Fact A-5. $$\left(1+\frac{1}{5}\right)^5 \quad < \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}.$$

(See "Review of Lectures – XI" page 6–7.) Similarly,

(6) $$\left(1+\frac{1}{6}\right)^6$$

$$= \quad 1 \qquad \text{(term 6-0)}$$

$$+ \frac{1}{1!} \cdot \frac{6}{6} \qquad \text{(term 6-1)}$$

$$+ \frac{1}{2!} \cdot \frac{6}{6} \cdot \frac{5}{6} \qquad \text{(term 6-2)}$$

$$+ \frac{1}{3!} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \qquad \text{(term 6-3)}$$

$$+ \frac{1}{4!} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \qquad \text{(term 6-4)}$$

$$+ \frac{1}{5!} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \qquad \text{(term 6-5)}$$

$$+ \frac{1}{6!} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6} \qquad \text{(term 6-6)}$$

What's clear is

Fact A-6. $$\left(1+\frac{1}{6}\right)^6 \quad < \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}.$$

On the other hand,

(7) $$\left(1+\frac{1}{7}\right)^7$$

$$
\begin{aligned}
= \quad & 1 && \text{(term 7-0)}\\
+ \; & \frac{1}{1!} \cdot \frac{7}{7} && \text{(term 7-1)}\\
+ \; & \frac{1}{2!} \cdot \frac{7}{7} \cdot \frac{6}{7} && \text{(term 7-2)}\\
+ \; & \frac{1}{3!} \cdot \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} && \text{(term 7-3)}\\
+ \; & \frac{1}{4!} \cdot \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} \cdot \frac{4}{7} && \text{(term 7-4)}\\
+ \; & \frac{1}{5!} \cdot \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} \cdot \frac{4}{7} \cdot \frac{3}{7} && \text{(term 7-5)}\\
+ \; & \frac{1}{6!} \cdot \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} \cdot \frac{4}{7} \cdot \frac{3}{7} \cdot \frac{2}{7} && \text{(term 7-6)}\\
+ \; & \big(\text{an extra term, which is positive}\big),
\end{aligned}
$$

(8) $$\left(1+\frac{1}{8}\right)^8$$

$$
\begin{aligned}
= \quad & 1 && \text{(term 8-0)}\\
+ \; & \frac{1}{1!} \cdot \frac{8}{8} && \text{(term 8-1)}\\
+ \; & \frac{1}{2!} \cdot \frac{8}{8} \cdot \frac{7}{8} && \text{(term 8-2)}\\
+ \; & \frac{1}{3!} \cdot \frac{8}{8} \cdot \frac{7}{8} \cdot \frac{6}{8} && \text{(term 8-3)}\\
+ \; & \frac{1}{4!} \cdot \frac{8}{8} \cdot \frac{7}{8} \cdot \frac{6}{8} \cdot \frac{5}{8} && \text{(term 8-4)}\\
+ \; & \frac{1}{5!} \cdot \frac{8}{8} \cdot \frac{7}{8} \cdot \frac{6}{8} \cdot \frac{5}{8} \cdot \frac{4}{8} && \text{(term 8-5)}\\
+ \; & \frac{1}{6!} \cdot \frac{8}{8} \cdot \frac{7}{8} \cdot \frac{6}{8} \cdot \frac{5}{8} \cdot \frac{4}{8} \cdot \frac{3}{8} && \text{(term 8-6)}\\
+ \; & \big(\text{extra terms, which are all positive}\big),
\end{aligned}
$$

(9) $$\left(1+\frac{1}{9}\right)^{9}$$

$$
\begin{aligned}
= \quad & 1 && \text{(term 9-0)}\\
+ \quad & \frac{1}{1!}\cdot\frac{9}{9} && \text{(term 9-1)}\\
+ \quad & \frac{1}{2!}\cdot\frac{9}{9}\cdot\frac{8}{9} && \text{(term 9-2)}\\
+ \quad & \frac{1}{3!}\cdot\frac{9}{9}\cdot\frac{8}{9}\cdot\frac{7}{9} && \text{(term 9-3)}\\
+ \quad & \frac{1}{4!}\cdot\frac{9}{9}\cdot\frac{8}{9}\cdot\frac{7}{9}\cdot\frac{6}{9} && \text{(term 9-4)}\\
+ \quad & \frac{1}{5!}\cdot\frac{9}{9}\cdot\frac{8}{9}\cdot\frac{7}{9}\cdot\frac{6}{9}\cdot\frac{5}{9} && \text{(term 9-5)}\\
+ \quad & \frac{1}{6!}\cdot\frac{9}{9}\cdot\frac{8}{9}\cdot\frac{7}{9}\cdot\frac{6}{9}\cdot\frac{5}{9}\cdot\frac{4}{9} && \text{(term 9-6)}\\
+ \quad & \bigl(\text{extra terms, which are all positive}\bigr),
\end{aligned}
$$

(10) $$\left(1+\frac{1}{10}\right)^{10}$$

$$
\begin{aligned}
= \quad & 1 && \text{(term 10-0)}\\
+ \quad & \frac{1}{1!}\cdot\frac{10}{10} && \text{(term 10-1)}\\
+ \quad & \frac{1}{2!}\cdot\frac{10}{10}\cdot\frac{9}{10} && \text{(term 10-2)}\\
+ \quad & \frac{1}{3!}\cdot\frac{10}{10}\cdot\frac{9}{10}\cdot\frac{8}{10} && \text{(term 10-3)}\\
+ \quad & \frac{1}{4!}\cdot\frac{10}{10}\cdot\frac{9}{10}\cdot\frac{8}{10}\cdot\frac{7}{10} && \text{(term 10-4)}\\
+ \quad & \frac{1}{5!}\cdot\frac{10}{10}\cdot\frac{9}{10}\cdot\frac{8}{10}\cdot\frac{7}{10}\cdot\frac{6}{10} && \text{(term 10-5)}\\
+ \quad & \frac{1}{6!}\cdot\frac{10}{10}\cdot\frac{9}{10}\cdot\frac{8}{10}\cdot\frac{7}{10}\cdot\frac{6}{10}\cdot\frac{5}{10} && \text{(term 10-6)}\\
+ \quad & \bigl(\text{extra terms, which are all positive}\bigr),
\end{aligned}
$$

and so on so forth. Here, as for the line "extra terms which are all positive" in each of part (7) through part (10), we don't need the precise form of that.

Now, let me pick up (term 6-2), (term 7-2), (term 8-2), (term 9-2) and (term 10-2):

$$\frac{1}{2!} \cdot \frac{6}{6} \cdot \frac{5}{6} \qquad \text{(term 6-2)}$$

$$\frac{1}{2!} \cdot \frac{7}{7} \cdot \frac{6}{7} \qquad \text{(term 7-2)}$$

$$\frac{1}{2!} \cdot \frac{8}{8} \cdot \frac{7}{8} \qquad \text{(term 8-2)}$$

$$\frac{1}{2!} \cdot \frac{9}{9} \cdot \frac{8}{9} \qquad \text{(term 9-2)}$$

$$\frac{1}{2!} \cdot \frac{10}{10} \cdot \frac{9}{10} \qquad \text{(term 10-2)}$$

- Each of these terms is smaller than $\frac{1}{2!}$. However, as you keep moving down in this list, the number $\big($for (term n-2), with $n = 100, 1000, ...\big)$ is getting closer and closer to $\frac{1}{2!}$ as n grows larger, because the values of

  $$\frac{5}{6}, \quad \frac{6}{7}, \quad \frac{7}{8}, \quad \frac{8}{9}, \quad \frac{9}{10}, \quad \ldots$$

  are getting closer and closer to 1.

Similarly, pick up (term 6-3), (term 7-3), (term 8-3), (term 9-3) and (term 10-3):

$$\frac{1}{3!} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \qquad \text{(term 6-3)}$$

$$\frac{1}{3!} \cdot \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} \qquad \text{(term 7-3)}$$

$$\frac{1}{3!} \cdot \frac{8}{8} \cdot \frac{7}{8} \cdot \frac{6}{8} \qquad \text{(term 8-3)}$$

$$\frac{1}{3!} \cdot \frac{9}{9} \cdot \frac{8}{9} \cdot \frac{7}{9} \qquad \text{(term 9-3)}$$

$$\frac{1}{3!} \cdot \frac{10}{10} \cdot \frac{9}{10} \cdot \frac{8}{10} \qquad \text{(term 10-3)}$$

- Each of these terms is smaller than $\dfrac{1}{3!}$. However, as you keep moving down in this list, the number $\big($for (term n-3), with $n = 100, 1000, \ldots\ \big)$ is getting closer and closer to $\dfrac{1}{3!}$ as n grows larger, because, in addition to the above, the values of

$$\frac{4}{6}, \quad \frac{5}{7}, \quad \frac{6}{8}, \quad \frac{7}{9}, \quad \frac{8}{10}, \quad \ldots$$

  are getting closer and closer to 1.

Now, the same thing is said about the other terms. Namely:

- Each of (term 6-4), (term 7-4), (term 8-4), (term 9-4), (term 10-4), *etc.* are smaller than $\dfrac{1}{4!}$. However, (term n-4) is getting closer and closer to $\dfrac{1}{4!}$ as n grows larger.

- Each of (term 6-5), (term 7-5), (term 8-5), (term 9-5), (term 10-5), *etc.* are smaller than $\dfrac{1}{5!}$. However, (term n-5) is getting closer and closer to $\dfrac{1}{5!}$ as n grows larger.

- Each of (term 6-6), (term 7-6), (term 8-6), (term 9-6), (term 10-6), *etc.* are smaller than $\dfrac{1}{6!}$. However, (term n-6) is getting closer and closer to $\dfrac{1}{6!}$ as n grows larger.

So, what does this entail? Yes, as n grows larger, eventually, the following happens:

As n grows larger, the sum of

- ○ the deficit of by how much (term n-2) is short of $\dfrac{1}{2!}$,
- ○ the deficit of by how much (term n-3) is short of $\dfrac{1}{3!}$,
- ○ the deficit of by how much (term n-4) is short of $\dfrac{1}{4!}$, and
- ○ the deficit of by how much (term n-5) is short of $\dfrac{1}{5!}$,

will get closer and closer to 0, whereas (term n-6) is at least

$$\frac{1}{6!} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6}$$

(which is just (term 6-6)) and it keeps growing. So ultimately, at some point, when n becomes large enough, the number $\left(1+\dfrac{1}{n}\right)^n$ exceeds the value

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}.$$

Hence we have proved the following:

Fact B-5. If you choose a large enough n, then

$$\left(1+\frac{1}{n}\right)^n > 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}.$$

- Extrapolation leads:

Fact B-6. If you choose a large enough n, then

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}.$$

Fact B-7. If you choose a large enough n, then

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!}.$$

Fact B-8. If you choose a large enough n, then

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}.$$

Fact B-9. If you choose a large enough n, then

$$\left(1+\frac{1}{n}\right)^n$$

$$> \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!}.$$

Fact B-10. If you choose a large enough n, then

$$\left(1+\frac{1}{n}\right)^n$$

$$> \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \frac{1}{10!}.$$

And so on and so forth. We can acutally compress all of thse into one single statement, which is as follows:

Fact B. Let k be an arbitrarily chosen positive integer, and fixed. By choosing a large enough n, we can make the following inequality true:

$$\boxed{\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}} \; .$$

⋆ Meanwhile, Fact A-5 and Fact A-6 $\big($on page 10$\big)$ can be generalized to

Fact A. Let k be an arbitrary positive integer. Then

$$\boxed{\left(1+\frac{1}{k}\right)^k \quad < \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}} \; .$$

So, what do these mean altogether? Yes, Fact A and Fact B mean precisely as follows:

Fact A and Fact B Compiled in one.

" *If you compare* $\boxed{\left(1+\dfrac{1}{k}\right)^k}$ and

$$\boxed{1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}} \; ,$$

for the same k, then always the latter is bigger, however, the latter

is exceeded by $\boxed{\left(1+\dfrac{1}{n}\right)^n}$ for a different, larger, n.

⋆ So, what does this *really* entail? Yes, by virtue of this, we have reached one important conclusion $\big($'Conclusion' below$\big)$.

Definition 1. The limit

$$\boxed{\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n}$$

means the 'threshold' number, the smallest number which $\left(1+\dfrac{1}{n}\right)^n$ cannot exceed when n runs through the entire positive integers.

Definition 2. The limit

$$\boxed{\lim_{k \to \infty} \left(1+\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}\right)}$$

means the 'threshold' number, the smallest number which

$$1+\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}$$

cannot exceed when k runs through the entire positive integers.

Conclusion. The above two limits are indeed equal.

$$\boxed{\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = \lim_{k \to \infty} \left(1+\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}\right)}\ .$$

- We denote this number as e. Thus

Definiiton 3 (The precise mathematical definition of e).

$$\boxed{\begin{aligned} e &= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \\ &= \lim_{k \to \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}\right). \end{aligned}}$$

- **Decimal expression of e.**

$$e = 2.71828182845904523536028747135266249775724709369 99...$$

Actually, if you care to do

$$\begin{array}{l} 1 + \frac{1}{1!}, \\ 1 + \frac{1}{1!} + \frac{1}{2!}, \\ 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}, \\ 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}, \\ 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}, \\ 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}, \\ 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!}, \\ \quad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \ddots \end{array}$$

in your calculator $\big($computer$\big)$, then

$$
\begin{aligned}
&1 + \frac{1}{1!} &&= 2, \\
&1 + \frac{1}{1!} + \frac{1}{2!} &&= 2.5, \\
&1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} &&= 2.6666666... \ , \\
&1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} &&= 2.7083333... \ , \\
&1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} &&= 2.7166666... \ , \\
&1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} &&= 2.7180555... \ , \\
&1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} &&= 2.7182539... \ , \\
&\qquad\vdots && \ \vdots
\end{aligned}
$$

As you keep moving, you quickly realize that the digits get stagnant. Actually, that might be a little hard to see, because your calculator can only show so many digits in the display screen $\big($eight, ten or twelve, or around that, depending on the model$\big)$, so you may not know what is really going on after the eighth, tenth or twelfth digit. But I can show you the following which I transcribed from my computer $\big($Maple$\big)$:

- The table of the decimal expression of the values

$$1+\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!}$$

for $\quad k = 1, 2, 3, 4, \ \cdots, 30 \quad \big($up to the 30th digit under the decimal point$\big)$:

$$
\begin{array}{rclcl}
k & = & 1 & \Longrightarrow & 2 \\
k & = & 2 & \Longrightarrow & 2.5 \\
k & = & 3 & \Longrightarrow & 2.666666666666666666666666666666... \\
k & = & 4 & \Longrightarrow & 2.708333333333333333333333333333... \\
k & = & 5 & \Longrightarrow & 2.716666666666666666666666666666... \\
k & = & 6 & \Longrightarrow & 2.718055555555555555555555555555... \\
k & = & 7 & \Longrightarrow & 2.718253968253968253968253968253... \\
k & = & 8 & \Longrightarrow & 2.718278769841269841269841269841... \\
k & = & 9 & \Longrightarrow & 2.718281525573192239858906525573... \\
k & = & 10 & \Longrightarrow & 2.718281801146384479717813051146... \\
k & = & 11 & \Longrightarrow & 2.718281826198492865159531826198... \\
k & = & 12 & \Longrightarrow & 2.718281828286168563946341724119... \\
k & = & 13 & \Longrightarrow & 2.718281828446759002314557870113... \\
k & = & 14 & \Longrightarrow & 2.718281828458229747912287594827... \\
k & = & 15 & \Longrightarrow & 2.718281828458994464285469576474... \\
k & = & 16 & \Longrightarrow & 2.718281828459042259058793450327... \\
k & = & 17 & \Longrightarrow & 2.718281828459045070516047795848... \\
k & = & 18 & \Longrightarrow & 2.718281828459045226708117481710... \\
k & = & 19 & \Longrightarrow & 2.718281828459045234928752728335... \\
k & = & 20 & \Longrightarrow & 2.718281828459045235339784490666... \\
k & = & 21 & \Longrightarrow & 2.718281828459045235359357431729... \\
k & = & 22 & \Longrightarrow & 2.718281828459045235360247110869... \\
k & = & 23 & \Longrightarrow & 2.718281828459045235360285792570... \\
k & = & 24 & \Longrightarrow & 2.718281828459045235360287404308... \\
k & = & 25 & \Longrightarrow & 2.718281828459045235360287468777... \\
k & = & 26 & \Longrightarrow & 2.718281828459045235360287471257... \\
k & = & 27 & \Longrightarrow & 2.718281828459045235360287471349... \\
k & = & 28 & \Longrightarrow & 2.718281828459045235360287471352... \\
k & = & 29 & \Longrightarrow & 2.718281828459045235360287471352... \\
k & = & 30 & \Longrightarrow & 2.718281828459045235360287471352...
\end{array}
$$

Now you have a better idea how the digits get stagnant as you move down on the list. As you can see, the last three are completely identical. But that's because you are only looking at the first 30 digits under the decimal point. If you look at the digits farther right, then you will find that those three numbers (the one with $k = 28$; the one with $k = 29$; and the one with $k = 30$) are actually different.

So what I am saying is, even beyond the 30th digit under the decimal point, the same is happening. If you look at the 100th, the 1000th, or the 10000th, digit under the decimal point, it will get stagnant if you move down on the list deep enough. No matter how far to the right from the decimal point it is that you are looking at, it will eventually get stagnant if you go down on the list deep enough.

So, e is the number with non-terminating decimal expression (meaning the decimal expression never stops), each of whose digits is the stagnated digit in the above sequence at the respective place under the decimal point.

This number e is so fundamental in mathematics. e shows up everywhere in mathematics. For example, e has a bearing on the theory of large factorials which I briefly alluded (the technical term: "asymptotic expansion"). I plan to cover that subject in due course.

In relation to the last lecture ("Review of Lectures – XI"), we used the metaphor you deposited a dollar in your bank account, and your bank offers an annual rate of 100 percent interest. If the compounding takes place continuously, then, after one year, the dollar amount of your balance is exactly e.

• **Appendix.** The following is only subsidiary inquiry. In view of Fact B in page 17, it makes sense to find the smallest positive integer n which makes the same inequality as in the statement of Fact B true. The smallest such n depends on k. I don't know if there is a simple formula for n as a function dependent on k, but for small k ($k = 2, 3, 4, 5, 6, 7, 8$ and 9) the answers are below (I relied on Maple software):

(2) The smallest positive integer n such that

$$\left(1+\frac{1}{n}\right)^n > 1 + \frac{1}{1!} + \frac{1}{2!}$$

is $n = 6$.

(3) The smallest positive integer n such that

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$$

is $n = 26$.

(4) The smallest positive integer n such that

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$$

is $n = 136$.

(5) The smallest positive integer n such that

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$$

is $n = 841$.

(6) The smallest positive integer n such that

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$$

is $n = 6006$.

(7) The smallest positive integer n such that

$$\left(1+\frac{1}{n}\right)^n \quad > \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!}$$

is $n = 48784$.

(8) The smallest positive integer n such that

$$\left(1+\frac{1}{n}\right)^n > 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}$$

is $n = 444364$.

(9) The smallest positive integer n such that

$$\left(1+\frac{1}{n}\right)^n$$

$$> 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+\frac{1}{9!}$$

is $n = 4487304$.