Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – XI (SUPPLEMENT)

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Appendix to §11. Football Series — "Half the distance to the goal'.

⋆ Out of the blue, how much is the each of the following?

(1) $\frac{1}{2} + \frac{1}{2} = ?$

(2) $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = ?$

(3) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = ?$

(4) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = ?$

(5) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} = ?$

(6) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64} = ?$

(7) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{128} = ?$

I will give you a clue. Do it one by one, as in do (1), (2), (3), (4), (5), (6) and (7) in this order. In each line, calculate the underlined portion first (below):

(1) $\underline{\frac{1}{2} + \frac{1}{2}} = ?$

(2) $\frac{1}{2} + \underline{\frac{1}{4} + \frac{1}{4}} = ?$

(3) $\frac{1}{2} + \frac{1}{4} + \underline{\frac{1}{8} + \frac{1}{8}} = ?$

(4) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \underline{\frac{1}{16} + \frac{1}{16}} = ?$

(5) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \underline{\frac{1}{32} + \frac{1}{32}} = ?$

(6) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \underline{\frac{1}{64} + \frac{1}{64}} = ?$

(7) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \underline{\frac{1}{128} + \frac{1}{128}} = ?$

Solutions:

(1) $\frac{1}{2} + \frac{1}{2}$ apparently equals 1.

(2) The underlined part is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. So part (2) is just $\frac{1}{2} + \frac{1}{2}$. This equals 1.

(3) The underlined part is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$. So part (3) is reduced to

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{4}.$$

This is exactly part (2). We have already calculated it and it equals 1.

(4) The underlined part is $\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$. So part (4) is reduced to

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}.$$

This is exactly part (3). We have already calculated it and it equals 1.

(5) The underlined part is $\frac{1}{32} + \frac{1}{32} = \frac{1}{16}$. So part (5) is reduced to

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}.$$

This is exactly part (4). We have already calculated it and it equals 1.

(6) The underlined part is $\frac{1}{64} + \frac{1}{64} = \frac{1}{32}$. So part (5) is reduced to

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32}.$$

This is exactly part (5). We have already calculated it and it equals 1.

(7) The underlined part is $\frac{1}{128} + \frac{1}{128} = \frac{1}{64}$. So part (5) is reduced to

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64}.$$

This is exactly part (6). We have already calculated it and it equals 1.

$\star$ Let's summarize the above:

(1) $$\frac{1}{2} + \frac{1}{2} = 1.$$

(2) $$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1.$$

(3) $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1.$$

(4) $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = 1.$$

(5) $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} = 1.$$

(6) $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64} = 1.$$

(7) $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{128} = 1.$$

Can you continue? Sure.

(8) $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{256} = 1.$$

(9) $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{512} = 1.$$

(10) $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \frac{1}{1024} = 1.$$

Paraphrase.

(1) $\frac{1}{2}$

is $\frac{1}{2}$ short of 1.

(2) $\frac{1}{2} + \frac{1}{4}$

is $\frac{1}{4}$ short of 1.

(3) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

is $\frac{1}{8}$ short of 1.

(4) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

is $\frac{1}{16}$ short of 1.

(5) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$

is $\frac{1}{32}$ short of 1.

(6) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$

is $\frac{1}{64}$ short of 1.

(7) $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \dfrac{1}{32} + \dfrac{1}{64} + \dfrac{1}{128}$

is $\dfrac{1}{128}$ short of 1.

(8) $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \dfrac{1}{32} + \dfrac{1}{64} + \dfrac{1}{128} + \dfrac{1}{256}$

is $\dfrac{1}{256}$ short of 1.

(9) $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \dfrac{1}{32} + \dfrac{1}{64} + \dfrac{1}{128} + \dfrac{1}{256} + \dfrac{1}{512}$

is $\dfrac{1}{512}$ short of 1.

(10) $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \dfrac{1}{32} + \dfrac{1}{64} + \dfrac{1}{128} + \dfrac{1}{256} + \dfrac{1}{512} + \dfrac{1}{1024}$

is $\dfrac{1}{1024}$ short of 1.

⋆ In the above, the denominators of the fractions in sight are all 2-to-the-powers:

$$\begin{aligned}
2^1 &= 2,\\
2^2 &= 4,\\
2^3 &= 8,\\
2^4 &= 16,\\
2^5 &= 32,\\
2^6 &= 64,\\
2^7 &= 128,\\
2^8 &= 256,\\
2^9 &= 512,\\
2^{10} &= 1024,\\
&\vdots \qquad\quad \vdots
\end{aligned}$$

Accrodingly, we can duplicate the above as

(1) $\frac{1}{2^1}$

is $\frac{1}{2^1}$ short of 1.

(2) $\frac{1}{2^1} + \frac{1}{2^2}$

is $\frac{1}{2^2}$ short of 1.

(3) $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3}$

is $\frac{1}{2^3}$ short of 1.

(4) $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$

is $\frac{1}{2^4}$ short of 1.

(5) $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5}$

is $\frac{1}{2^5}$ short of 1.

(6) $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}$

is $\frac{1}{2^6}$ short of 1.

(6) $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7}$

is $\frac{1}{2^7}$ short of 1.

(8) $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8}$

is $\frac{1}{2^8}$ short of 1.

(9) $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9}$

is $\frac{1}{2^9}$ short of 1.

(10) $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \frac{1}{2^{10}}$

is $\frac{1}{2^{10}}$ short of 1.

⋆ By extrapolating, we conclude

Formula.

$$\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n} \quad \text{is} \quad \frac{1}{2^n} \quad \text{short of 1.}$$

Formula paraphrased.

$$\boxed{\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.}$$

Exercise. (a) How much is the following quantity?

$$(20)\qquad \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}+\frac{1}{2^8}+\frac{1}{2^9}+\frac{1}{2^{10}}$$
$$+\frac{1}{2^{11}}+\frac{1}{2^{12}}+\frac{1}{2^{13}}+\frac{1}{2^{14}}+\frac{1}{2^{15}}+\frac{1}{2^{16}}+\frac{1}{2^{17}}+\frac{1}{2^{18}}+\frac{1}{2^{19}}+\frac{1}{2^{20}}.$$

(b) Is the above quantity less than 1, greater than 1, or equal to 1?

$\Big[$ **Answer** $\Big]$**:** (a) $1-\frac{1}{2^{20}}$. (b) Less than 1.

$\star$ I can use the following model to explain what the above formula says:

Metaphor. In a football game, suppose the ball is placed at the 50 yard line.

Then half the distance to the goal line is one half of 50 yards, which is $\frac{50}{2}=25$ yards. Suppose you gain precisely that yardage. So in the next play, the ball is placed at the $\frac{50}{2}=25$ yard line.

Then half the distance to the goal line is one half of $\frac{50}{2}=25$ yards, which is $\frac{50}{4}=12.5$ yards. Suppose you gain precisely that yardage. So in the next play, the ball is placed at the $\frac{50}{4}=12.5$ yard line.

Then half the distance to the goal line is one half of $\frac{50}{4}=12.5$ yards, which is $\frac{50}{8}=6.25$ yards. Suppose you gain precisely that yardage. So in the next play, the ball is placed at the $\frac{50}{8}=6.25$ yard line.

Then half the distance to the goal line is one half of $\frac{50}{8}=6.25$ yards, which is $\frac{50}{16}=3.125$ yards. Suppose you gain precisely that yardage. So in the next play, the ball is placed at the $\frac{50}{16}=3.125$ yard line, and so on so forth.

No matter how many times you keep gaining half the distance to the goal line, the ball never reaches the goal line.