Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XI

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§11. *e*. Continued.

- Last time I claimed the following at the end:

Claim. The numbers $\left(1+\dfrac{1}{n}\right)^n$; $n = 1, 2, 3, 4, \cdots,$ cannot become arbitrarily large. Indeed, the digit before the decimal point in the decimal expression of each of these numbers is always 2, no matter how large n is. In other words, these numbers are all between 2 and 3.

Today I am going to give a mathematical reasoning why the above claim is true. But first I am going to tell you that something like this naturally meshes well with some real life example. Before everything, in the following 'Metaphor', we relax the smallest currency unit, meaning: In reality, we cannot divide one cent. But here we work on a model where one can divide any dollar amount by any large number (integer). Also, we never round figures. So, one-third of a dollar is never the same as 33 cents (because 33 cents is one-third of 99 cents).

Metaphor. Now, you open a bank account, deposit a dollar in that account. Your bank offers 10 percent interest annually. After one year, your balance is a dollar and ten cents. But suppose another bank offers 100 percent interest annually. Then you probably want to forget the first bank and rush to the second bank, right? So you actually went to the second bank, with 100 percent annual interest rate. There you opened a bank account, and deposited a dollar in that account. Then after one year your balance is two dollars. A much better deal.

Intermission. Since the interest calculation with the second bank is easier so let's stick with the second bank. Indeed, as it turns out, with any interest rate the gist of what I'm going to lay out is the same. The difference is a constant multiplication in the exponent. Having or not having that constant is mathematically insubstantial. With the 100 percent of the rate we can make that constant 1, or in other words, that way we can get rid of the subsidiary constant. The general case is easily reconstructed from that special case with an obvious tweak. So let's forget the first bank with 10 percent interest, and stick with the second bank with 100 percent interest.

Metaphor continued. Now, in the second bank, with 100 percent interest, suppose you don't mess with your account, as in once you have opened the account and made a dollar of a deposit, you never withdraw money, or make additional deposit. You just let your money sit there. Like I said, after the first year, the balance is two dollars, of which one dollar is accrued as an interest. After the second year, should the balance be three dollars, or four dollars?

If you say three dollars, then that's correct, because the annual interest rate is 100 percent, and your deposit was one dollar, so besides the interest accrued after the first year period, which is a dollar, another dollar was accrued as an interest by virtue of the fact that that original one dollar deposit remained to sit throughout the second year.

But some of you might say no, four dollars, because at the end of the first year your deposit grew into two dollars. 100 percent of two dollars should be added to your balance as the interest for the second year. If you say so, you are correct too.

So, both are correct. Namely, it depends on whether your bank says your accrued interest is included or not as the base of calculating the interest for the next period. In other words, it depends on whether your banks says the interest is compounded.

Another Intermission. Actually, my take is, if my bank has a non-compounded interest system, then I would withdraw all two dollars of my money after one year, close my account, and then open a brand new account and make a deposit of two dollars. Then my bank has to base two dollars in calculating the interest for the next period, so this has the same effect as the compounded interest (assuming that there is no fee in opening and closing accounts). So, in this sense, banks should offer a compound interest or they would have to deal with the never-ending demands of closing and re-opening of accounts, which is impractical.

Metaphor continued. On that basis the second bank actually offers a compound interest with 100 percent interest rate annually. But then there is a third bank, a competitor, that advertises as follows: They offer the same interest rate of 100 percent annually, but they calculate the intest more frequently than once a year, namely, twice a year.

Don't be misled: What the third bank is *not* saying is they offer 100 percent interest semi-anunally, so your money would grow like after six months what was originally a dollar would grow into two dollars, and then after another six months that two dollars would further grow into four dollars, and so on. That's *not* what they are advertising. They assessed their financial competency and figured out that they would bankrupt if they did that. But rather, they can still afford to offer the following deal: They keep the 100 percent annual interest rate, but the 100 percent annual rate translates to 50 percent semi-annual interest rate. But if compounding takes place semi-annually with that rate, that's a better deal than the 100 percent annual interest rate with compounding taking place just annually. Are you following me? Let's mathematically dissect.

- With the second bank $\big($annual interest rate is 100 percent, compounded annually$\big)$, after 1 year your balance is

$$\$\left(1+1\right).$$

- With the third bank $\big($annual interest rate is 100 percent, compounded semi-annually$\big)$, after $\dfrac{1}{2}$ year your balance is

$$\$\left(1+\frac{1}{2}\right),$$

and after 1 year it is

$$\$\left(1+\frac{1}{2}\right)^2.$$

Now, there is a fourth bank, that tries to outplay the third bank, and they advertise the 100 percent annual interest rate, which itself is the same, but they compound the interest three times a year, each time applying $\dfrac{1}{3}$ of 100 percent rate. Then

- With the fourth bank, your balance after $\dfrac{1}{3}$ year is

$$\$\left(1+\frac{1}{3}\right),$$

after $\dfrac{2}{3}$ year it is

$$\$ \left(1+\frac{1}{3}\right)^2,$$

and after 1 year it is

$$\$ \left(1+\frac{1}{3}\right)^3.$$

Now, there is a fifth bank, that tries to outplay the fourth bank, and they advertise the 100 percent annual interest rate, which itself is the same, but they compound the interest quarter-annually $\big($four times a year$\big)$, each time applies $\dfrac{1}{4}$ of 100 percent interest rate. Then

- With the fifth bank, your balance after $\dfrac{1}{4}$ year is

$$\$ \left(1+\frac{1}{4}\right),$$

after $\dfrac{2}{4}$ year it is

$$\$ \left(1+\frac{1}{4}\right)^2,$$

after $\dfrac{3}{4}$ year it is

$$\$ \left(1+\frac{1}{4}\right)^3,$$

and after 1 year it is

$$\$ \left(1+\frac{1}{4}\right)^4.$$

And so on so forth. Do you see the picture here? Just focus on the balance after one year, in each scenario $\big($with each of the second through the fifth banks$\big)$. With a dollar of a deposit, with 100 percent annual interest rate, and the compounding takes place n times a year, with $n = 1, 2, 3$ and 4, the balance is

$$\$\left(1+1\right)^{1},$$

$$\$\left(1+\frac{1}{2}\right)^{2},$$

$$\$\left(1+\frac{1}{3}\right)^{3},$$

and

$$\$\left(1+\frac{1}{4}\right)^{4},$$

respectively. We have mathematically verified last time that this is an increasing sequence. What that translates to is that, the higher the compounding frequency is, the more additional benefit you receive. Now, that is consistent with our intuition. But here, the real question is, as we increase the compounding frequency, does the benefit increase unlimitedly? As in can one dollar grow into one million dollars, one billion dollars, or one trillion dollars, by setting the compounding frequency to be certain time-length, such as 'nanosecond' $\left(= \frac{1}{10^9} \text{ seconds}\right)$?

What do you think? My answer is 'no'. The additional gain will ultimately diminish as you keep increasing the compounding frequency. No matter how you set that frequency, to be 'nanosecond' or shorter, the balance after 1 year is a little less than three dollars. How come? This is exactly the claim that I showed you right at the beginning of today's class. Can you mathematically substantiate that claim? That's what we are going to do for the rest of today's class.

So, we refer to the last lecture. What we have work out is that $\left(1+\frac{1}{5}\right)^{5}$ equals the following quantity:

$$
\begin{aligned}
&1 \\
&+ \frac{1}{1} \cdot \underbrace{\frac{5}{5}}_{\textcircled{1}} \\
&+ \frac{1}{1 \cdot 2} \cdot \underbrace{\frac{5}{5} \cdot \frac{4}{5}}_{\textcircled{2}} \\
&+ \frac{1}{1 \cdot 2 \cdot 3} \cdot \underbrace{\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5}}_{\textcircled{3}} \\
&+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \underbrace{\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5}}_{\textcircled{4}} \\
&+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \underbrace{\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5}}_{\textcircled{5}} .
\end{aligned}
$$

But if you look at the portion underlined, they are all less than 1, except ① equals 1:

$$
\begin{aligned}
\frac{5}{5} &= 1, \\
\frac{5}{5} \cdot \frac{4}{5} &< 1, \\
\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} &< 1,
\end{aligned}
$$

$$\frac{5}{5}\cdot\frac{4}{5}\cdot\frac{3}{5}\cdot\frac{2}{5} < 1,$$

$$\frac{5}{5}\cdot\frac{4}{5}\cdot\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{1}{5} < 1$$

(see "Review of Lectures – X Supplement"). So, if you replace the underlined parts with 1, then the resulting quantity becomes bigger (once again, see "Review of Lectures – X Supplement"). In short,

$$\left(1+\frac{1}{5}\right)^5$$

$$< 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}.$$

Now, if you further compare this latter quantity with

$$1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 2} + \frac{1}{1\cdot 2\cdot 2\cdot 2} + \frac{1}{1\cdot 2\cdot 2\cdot 2\cdot 2}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4},$$

then this last quantity $1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}$ is clearly bigger, because

$$\frac{1}{1\cdot 2\cdot 3} < \frac{1}{1\cdot 2\cdot 2},$$
$$\frac{1}{1\cdot 2\cdot 3\cdot 4} < \frac{1}{1\cdot 2\cdot 2\cdot 2}, \qquad \text{and}$$
$$\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} < \frac{1}{1\cdot 2\cdot 2\cdot 2\cdot 2}.$$

Now, we know the fact that this last quantity $1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}$ is $\frac{1}{2^4}$ short of 3 (see "Supplement").

So, in short,

$$\left(1+\frac{1}{5}\right)^5 \quad < \quad 3.$$

The same argument works for

$$(n) \qquad \left(1+\frac{1}{n}\right)^n ,$$

with any n. In sum, we draw the following conclusion:

Conclusion. For an arbitrary positive integer $n = 1, 2, 3, 4, \cdots$,

$$\boxed{\left(1+\frac{1}{n}\right)^n \quad < \quad 3.}$$

- The next question is to identify the 'limit' of these numbers. Namely, we are going to figure out the balance after one year with "continuous" compounding, namely, the frequency of compounding n approaches to infinity.