Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – X (SUPPLEMENT)

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APPENDIX TO §10.   FRACTIONS AND INEQUALITIES REFRESHERS.

- $\boxed{\textbf{Fractions — Refresher \#1}}$ •   First, let's agree

$$\frac{1}{1} \;=\; 1.$$

Next, agree

$$1 - \frac{1}{2} \;=\; \frac{1}{2},$$

$$1 - \frac{1}{3} \;=\; \frac{2}{3},$$

$$1 - \frac{1}{4} \;=\; \frac{3}{4},$$

$$1 - \frac{1}{5} \;=\; \frac{4}{5},$$

$$1 - \frac{1}{6} \;=\; \frac{5}{6},$$

$$\vdots$$

More generally,

$$\boxed{\; 1 - \frac{1}{n} \;=\; \frac{n-1}{n} \;}\; .$$

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- **Fractions — Refresher #2**   Agree with the following:

$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2 \cdot 3},$$

$$\frac{1}{4} \cdot \frac{1}{6} \cdot \frac{1}{7} = \frac{1}{4 \cdot 6 \cdot 7},$$

$$\frac{1}{5} \cdot \frac{1}{9} \cdot \frac{1}{10} \cdot \frac{1}{12} = \frac{1}{5 \cdot 9 \cdot 10 \cdot 12}.$$

More generally:

$$\boxed{\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}}, \quad \boxed{\frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c} = \frac{1}{abc}}, \quad \boxed{\frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c} \cdot \frac{1}{d} = \frac{1}{abcd}},$$

and so on. If you use these rules,

$$\left(\frac{1}{2}\right)^2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2 \cdot 2},$$

$$\left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2 \cdot 2 \cdot 2},$$

$$\left(\frac{1}{2}\right)^4 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2 \cdot 2 \cdot 2 \cdot 2},$$

- The same for a instead of 2.

$$\left(\frac{1}{a}\right)^2 = \frac{1}{a} \cdot \frac{1}{a} = \frac{1}{a \cdot a},$$

$$\left(\frac{1}{a}\right)^3 = \frac{1}{a} \cdot \frac{1}{a} \cdot \frac{1}{a} = \frac{1}{a \cdot a \cdot a},$$

$$\left(\frac{1}{a}\right)^4 = \frac{1}{a} \cdot \frac{1}{a} \cdot \frac{1}{a} \cdot \frac{1}{a} = \frac{1}{a \cdot a \cdot a \cdot a},$$

$$\vdots$$

- $\boxed{\textbf{Fractions} \quad — \quad \textbf{Refresher \#3}}$ Observe

$$\frac{3}{5} \cdot \frac{1}{8} \;=\; \frac{3}{5 \cdot 8}, \qquad \text{and}$$

$$\frac{1}{5} \cdot \frac{3}{8} \;=\; \frac{3}{5 \cdot 8}.$$

So, these are equal. More generally,

$$\boxed{\;\frac{a}{b} \cdot \frac{1}{c} \;=\; \frac{1}{b} \cdot \frac{a}{c}\;} \;\;.$$

- $\boxed{\textbf{Fractions} \quad — \quad \textbf{Refresher \#4}}$ Observe

$$\frac{3 \cdot 2}{4 \cdot 5} \;=\; \frac{3}{4} \cdot \frac{2}{5},$$

$$\frac{7 \cdot 5 \cdot 4}{6 \cdot 2 \cdot 2} \;=\; \frac{7}{6} \cdot \frac{5}{2} \cdot \frac{4}{2}, \qquad \text{and}$$

$$\frac{8 \cdot 5 \cdot 6 \cdot 1}{9 \cdot 8 \cdot 4 \cdot 6} \;=\; \frac{8}{9} \cdot \frac{5}{8} \cdot \frac{6}{4} \cdot \frac{1}{6}.$$

More generally,

$$\boxed{\;\frac{a\,b}{a'b'} \;=\; \frac{a}{a'} \cdot \frac{b}{b'}\;} \;\;,$$

$$\boxed{\;\frac{a\,b\,c}{a'b'c'} \;=\; \frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'}\;} \;\;,$$

$$\boxed{\;\frac{a\,b\,c\,d}{a'b'c'd'} \;=\; \frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'} \cdot \frac{d}{d'}\;} \;\;,$$

and so on.

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- The following is a little more involved, but it's just some combinations of the above:

$$\binom{5}{4} \cdot \left(\frac{1}{5}\right)^4 \; = \; \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5}$$

$$= \; \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{1}{5 \cdot 5 \cdot 5 \cdot 5}$$

$$= \; \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{5 \cdot 4 \cdot 3 \cdot 2}{5 \cdot 5 \cdot 5 \cdot 5}$$

$$= \; \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5}.$$

In short,

$$\binom{5}{4} \cdot \left(\frac{1}{5}\right)^4 \; = \; \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5}.$$

Exercise 1. Write each of the following quantity in the similar way.

$$\binom{4}{2} \cdot \left(\frac{1}{4}\right)^2. \qquad \binom{5}{3} \cdot \left(\frac{1}{5}\right)^3. \qquad \binom{6}{5} \cdot \left(\frac{1}{6}\right)^5.$$

$\boxed{\textbf{Answers}}$:
$$\binom{4}{2} \cdot \left(\frac{1}{4}\right)^2 \; = \; \frac{1}{1 \cdot 2} \cdot \frac{4}{4} \cdot \frac{3}{4},$$

$$\binom{5}{3} \cdot \left(\frac{1}{5}\right)^3 \; = \; \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5},$$

$$\binom{6}{5} \cdot \left(\frac{1}{6}\right)^5 \; = \; \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6}.$$

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- $\boxed{\textbf{Inequalities} \quad - \quad \textbf{Refresher \#1}}$ (1)   When I write

$$\boxed{a \quad < \quad b} \, ,$$

this reads

" $\underline{\underline{a \text{ is smaller (less) than } b.}}$ "

Alternatively,

" $\underline{\underline{b \text{ is bigger (larger) than } a.}}$ "

(2)   Also, when I write

$$\boxed{c \quad > \quad d} \, ,$$

this reads

" $\underline{\underline{c \text{ is bigger (larger) than } d.}}$ "

Alternatively,

" $\underline{\underline{d \text{ is smaller (less) than } c.}}$ "

(3)   Next, when I write

$$\boxed{a \quad > \quad 0} \, ,$$

this reads

" $\underline{\underline{a \text{ is bigger (larger) than } 0.}}$ "

Alternatively,

" $\underline{\underline{a \text{ is positive.}}}$ "

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(4)   Also, when I write

$$\boxed{b \;\; < \;\; 0}\;,$$

this reads

" $\underline{\underline{b \;\; \text{is smaller} \;\; (\text{less}) \;\; \text{than} \;\; 0.}}$ "

Alternatively,

" $\underline{\underline{b \;\; \text{is negative.}}}$ "

- $\boxed{\textbf{Inequalities} \;\; — \;\; \textbf{Refresher \#2}}$ Agree with the following:

Fact. $\underline{\underline{\text{If}}}$ $\boxed{a \;\; < \;\; b}$ $\underline{\underline{\text{and moreover} \;\; a \;\; \text{is positive, then}}}$

$$\boxed{\dfrac{1}{a} \;\; > \;\; \dfrac{1}{b}}\;.$$

★   This reflects the reality that, if you are to share a pizza with more people, then your share becomes smaller.

Example.   We all know

$$1 \; < \; 2, \qquad 2 \; < \; 3, \qquad 3 \; < \; 4, \qquad 4 \; < \; 5, \qquad \text{and} \qquad 5 \; < \; 6.$$

From these, we also know

$$\frac{1}{1} \; > \; \frac{1}{2}, \qquad \frac{1}{2} \; > \; \frac{1}{3}, \qquad \frac{1}{3} \; > \; \frac{1}{4}, \qquad \frac{1}{4} \; > \; \frac{1}{5} \quad \text{and} \quad \frac{1}{5} \; > \; \frac{1}{6}.$$

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- **Inequalities — Refresher #3**   Agree with the following:

Facts.

  ◦   <u>If</u> $\boxed{\ a\ <\ b\ }$ <u>then</u> $\boxed{\ 2\,a\ <\ 2\,b\ }$,

  ◦   <u>If</u> $\boxed{\ a\ <\ b\ }$ <u>then</u> $\boxed{\ 3\,a\ <\ 3\,b\ }$,

  ◦   <u>If</u> $\boxed{\ a\ <\ b\ }$ <u>then</u> $\boxed{\ 4\,a\ <\ 4\,b\ }$,

  ◦   <u>If</u> $\boxed{\ a\ <\ b\ }$ <u>then</u> $\boxed{\ 5\,a\ <\ 5\,b\ }$,

$$\vdots \qquad\qquad \vdots$$

★ More generally:

Fact.   <u>If</u> $\boxed{\ a < b\ }$ <u>and</u> $\boxed{\ t > 0\ }$ <u>then</u> $\boxed{\ t\,a\ <\ t\,b\ }$.

Example.   From

$$\frac{1}{3} > \frac{1}{4}, \qquad\qquad \frac{1}{4} > \frac{1}{5} \qquad \text{and} \qquad \frac{1}{5} > \frac{1}{6},$$

we know

$$2 \cdot \frac{1}{3} > 2 \cdot \frac{1}{4}, \qquad\qquad 3 \cdot \frac{1}{4} > 3 \cdot \frac{1}{5} \qquad \text{and} \qquad 4 \cdot \frac{1}{5} > 4 \cdot \frac{1}{6}.$$

In other words:

$$\frac{2}{3} > \frac{2}{4}, \qquad\qquad \frac{3}{4} > \frac{3}{5} \qquad \text{and} \qquad \frac{4}{5} > \frac{4}{6}.$$

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- $\boxed{\textbf{Inequalities } \text{---} \textbf{ Refresher \#4}}$ Agree with the following:

$$\underline{\underline{\text{If}}} \quad \boxed{c \; > \; d} \quad \underline{\underline{\text{then}}} \quad \boxed{1 - c \; < \; 1 - d} \; .$$

$\star$ This reflects the following reality: You give a dollar bill at the casher to buy either item A, or item B. Then the change is smaller if you end up buying whichever item is pricier.

Example.   From

$$\frac{1}{1} \; > \; \frac{1}{2}, \qquad \frac{1}{2} \; > \; \frac{1}{3}, \qquad \frac{1}{3} \; > \; \frac{1}{4}, \qquad \frac{1}{4} \; > \; \frac{1}{5} \quad \text{and} \quad \frac{1}{5} \; > \; \frac{1}{6},$$

we know

$$1 - \frac{1}{1} \; < \; 1 - \frac{1}{2},$$

$$1 - \frac{1}{2} \; < \; 1 - \frac{1}{3},$$

$$1 - \frac{1}{3} \; < \; 1 - \frac{1}{4},$$

$$1 - \frac{1}{4} \; < \; 1 - \frac{1}{5}, \qquad \text{and}$$

$$1 - \frac{1}{5} \; < \; 1 - \frac{1}{6}.$$

$\star$ Paraphrase:

$$0 \; < \; \frac{1}{2}, \qquad \frac{1}{2} \; < \; \frac{2}{3}, \qquad \frac{2}{3} \; < \; \frac{3}{4}, \qquad \frac{3}{4} \; < \; \frac{4}{5} \quad \text{and} \quad \frac{4}{5} \; < \; \frac{5}{6}.$$

Example. From

$$\frac{2}{3} > \frac{2}{4}, \qquad \frac{2}{4} > \frac{2}{5} \quad \text{and} \quad \frac{2}{5} > \frac{2}{6},$$

we know

$$1 - \frac{2}{3} < 1 - \frac{2}{4},$$

$$1 - \frac{2}{4} < 1 - \frac{2}{5}, \qquad \text{and}$$

$$1 - \frac{2}{5} < 1 - \frac{2}{6}.$$

⋆ Paraphrase:

$$\frac{1}{3} < \frac{2}{4}, \qquad \frac{2}{4} < \frac{3}{5} \quad \text{and} \quad \frac{3}{5} < \frac{4}{6}.$$

Example. From

$$\frac{3}{4} > \frac{3}{5} \qquad \text{and} \qquad \frac{3}{5} > \frac{3}{6},$$

we know

$$1 - \frac{3}{4} < 1 - \frac{3}{5}, \qquad \text{and}$$

$$1 - \frac{3}{5} < 1 - \frac{3}{6}.$$

⋆ Paraphrase:

$$\frac{1}{4} < \frac{2}{5}, \qquad \text{and} \qquad \frac{2}{5} < \frac{3}{6}.$$

- $\boxed{\textbf{Inequalities — Refresher \#5}}$ Finally, agree with the following:

Fact.

$\underline{\underline{\text{Suppose all of } a, \ b, \ c, \ d \ \text{are positive.}}}$

$\underline{\underline{\text{If}}}$ $\boxed{a < b}$ $\underline{\underline{\text{and}}}$ $\boxed{c < d}$ $\underline{\underline{\text{then}}}$ $\boxed{a\,c \ < \ b\,d}$.

- Now it should be easy to see why each of the following is true:

$$\frac{5}{5} \cdot \frac{4}{5} < \frac{6}{6} \cdot \frac{5}{6}, \qquad \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} < \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6},$$

$$\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} < \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6}, \qquad \text{and}$$

$$\frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5} < \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6}.$$

These will be used in the main text $\big(\text{"Review of Lectures – X", page 8–9}\big).$