Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – X

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§10. *e*. Intro.

• Today's class involves some fractions. So, be ready. Now, you probably know by now that, when I throw something out of no context, that's a warm-up for something else which is 'relevant'. So, "not for nothing", let's "compare"

(1) $$\left(1+\frac{1}{1}\right)^{1},$$

(2) $$\left(1+\frac{1}{2}\right)^{2},$$

(3) $$\left(1+\frac{1}{3}\right)^{3},$$

(4) $$\left(1+\frac{1}{4}\right)^{4},$$

(5) $$\left(1+\frac{1}{5}\right)^{5},$$

$$\vdots$$

In decimals, these are

(1) $$2\ ,$$

(2) $$1.5 \cdot 1.5 \quad ,$$

(3) $$(1.333333...) \cdot (1.333333...) \cdot (1.333333...) \quad ,$$

(4) $$1.25 \cdot 1.25 \cdot 1.25 \cdot 1.25 \quad , \qquad \text{and}$$

(5) $$1.2 \cdot 1.2 \cdot 1.2 \cdot 1.2 \cdot 1.2 \quad .$$

Can you tell at first glance which one is bigger, part (1) or part (2)? Probably part (2). But then, which one is bigger, part (2) or part (3)? Or, which one is bigger, part (3) or part (4)? *etc.*

You might say "hey, use calculators, silly". Yes I know. So, I might as well just say "why not", urge everyone to calculate those five numbers using calculators, compare the figures and decide which one's the biggest, which one's the smallest, *etc.* and quit after five minutes, and that will be a happy ending. That's too predictable and not interesting. What's interesting is, I am actually going to tell you something that will ultimately make you think twice about an indiscriminate use of calculators. For that matter, let's first recall

$$
\begin{aligned}
10^2 &= 10 \cdot 10 = 100 \ \text{(one hundred)}. \\
10^3 &= 10 \cdot 10 \cdot 10 = 1000 \ \text{(one thousand)}. \\
10^4 &= 10 \cdot 10 \cdot 10 \cdot 10 = 10000 \ \text{(ten thousand)}. \\
10^5 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100000 \ \text{(one hundred thousand)}. \\
10^6 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1000000 \ \text{(one million)}. \\
10^7 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000000 \ \text{(ten million)}. \\
10^8 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100000000 \ \text{(one hundred million)}. \\
10^9 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1000000000 \ \text{(one billion)}. \\
10^{10} &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000000000 \\
&\qquad \text{(ten billion)}. \\
10^{11} &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100000000000 \\
&\qquad \text{(one hundred billion)}. \\
10^{12} &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1000000000000 \\
&\qquad \text{(one trillion)}. \\
\vdots & \qquad \vdots \qquad \ddots
\end{aligned}
$$

- More generally, for a positive integer n,

$$
\boxed{10^n = 1\underbrace{00000\ \ldots\ 0}_{n}} .
$$

Now, what if I say

"*Okay, class: Use your calculator and compare the numbers below.*"

$$(10^3) \qquad \left(1+\frac{1}{10^3}\right)^{10^3},$$

$$(10^6) \qquad \left(1+\frac{1}{10^6}\right)^{10^6},$$

$$(10^9) \qquad \left(1+\frac{1}{10^9}\right)^{10^9},$$

$$(10^{12}) \qquad \left(1+\frac{1}{10^{12}}\right)^{10^{12}},$$

$$(10^{15}) \qquad \left(1+\frac{1}{10^{15}}\right)^{10^{15}},$$

$$(10^{18}) \qquad \left(1+\frac{1}{10^{18}}\right)^{10^{18}},$$

$$\vdots$$

Now, this depends on the model of your calculator, but I know one of the models gives

(10^3)	2.7169..,
(10^6)	2.7182804..,
(10^9)	2.718281827..,
(10^{12})	2.718281828..,
(10^{15})	1,
(10^{18})	1,

(In other models too '1' starts to show up, though where exactly depends.) Do you think these are accurate, though? I say 'no'. Some of these are not accurate. Namely, part (10^{15}) and part (10^{18}) are way off the mark. Part (10^{15}) should be a tiny bit bigger than part (10^{12}); part (10^{18}) should be a tiny tiny bit bigger than part (10^{15}), and so on. And this trend continues for forever. Your calculator is totally lying. How can I say that? I'm one hundred twenty percent sure I know what I am talking about. As in I can bet my money on it. So, wanna bet? To give you some nuts-and-bolts, your calculator rounds numbers where it shouldn't, and sometimes that leads to an error. Do you believe me, or calculator? I can give you an absolutely irrefutable logic to discredit some of what your calculator spits out in this particular case.

For that, it is beneficial to go back to

(1) $$\left(1+\frac{1}{1}\right)^{1},$$

(2) $$\left(1+\frac{1}{2}\right)^{2},$$

(3) $$\left(1+\frac{1}{3}\right)^{3},$$

(4) $$\left(1+\frac{1}{4}\right)^{4},$$

(5) $$\left(1+\frac{1}{5}\right)^{5},$$

(6) $$\left(1+\frac{1}{6}\right)^{6},$$

$$\vdots$$

This sequence continues endlessly. Agree that (10^{12}), (10^{15}), (10^{18}), *etc.* are sitting in this sequence. It is just that they are very very far down. Now, I am going to give you a proof of the fact that part (6) is bigger than part (5), without relying on calculator, or without directly hand-calculating the values of part (5) and part (6) each.

This is where Binomial Formula (Formula B in “Review of Lectures – VII”, page 13) comes in handy. I want to rewrite part (5) as follows:

$$
\begin{aligned}
(5) \qquad \left(1+\frac{1}{5}\right)^5 \quad = \quad & \binom{5}{0}\cdot 1^5 \\
&+ \binom{5}{1}\cdot 1^4 \cdot \left(\frac{1}{5}\right) \\
&+ \binom{5}{2}\cdot 1^3 \cdot \left(\frac{1}{5}\right)^2 \\
&+ \binom{5}{3}\cdot 1^2 \cdot \left(\frac{1}{5}\right)^3 \\
&+ \binom{5}{4}\cdot 1 \cdot \left(\frac{1}{5}\right)^4 \\
&+ \binom{5}{5} \cdot \left(\frac{1}{5}\right)^5
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 1 \\
&+ \frac{5}{1}\cdot\frac{1}{5} \\
&+ \frac{5\cdot 4}{1\cdot 2}\cdot\frac{1}{5\cdot 5} \\
&+ \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}\cdot\frac{1}{5\cdot 5\cdot 5} \\
&+ \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{5\cdot 5\cdot 5\cdot 5} \\
&+ \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{1}{5\cdot 5\cdot 5\cdot 5\cdot 5}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 1 \\
& + \frac{1}{1} \cdot \frac{5}{5} \\
& + \frac{1}{1 \cdot 2} \cdot \frac{5 \cdot 4}{5 \cdot 5} \\
& + \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{5 \cdot 4 \cdot 3}{5 \cdot 5 \cdot 5} \\
& + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{5 \cdot 4 \cdot 3 \cdot 2}{5 \cdot 5 \cdot 5 \cdot 5} \\
& + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 1 \\
& + \frac{1}{1} \cdot \frac{5}{5} \\
& + \frac{1}{1 \cdot 2} \cdot \frac{5}{5} \cdot \frac{4}{5} \\
& + \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \\
& + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \\
& + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5} .
\end{aligned}
$$

In short,part (5) equals

$$1 \qquad \text{(term 5-0)}$$

$$+\ \frac{1}{1} \cdot \frac{5}{5} \qquad \text{(term 5-1)}$$

$$+\ \frac{1}{1 \cdot 2} \cdot \frac{5}{5} \cdot \frac{4}{5} \qquad \text{(term 5-2)}$$

$$+\ \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \qquad \text{(term 5-3)}$$

$$+\ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \qquad \text{(term 5-4)}$$

$$+\ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5}\,. \qquad \text{(term 5-5)}$$

Similarly, part (6) equals

$$1 \qquad \text{(term 6-0)}$$

$$+\ \frac{1}{1} \cdot \frac{6}{6} \qquad \text{(term 6-1)}$$

$$+\ \frac{1}{1 \cdot 2} \cdot \frac{6}{6} \cdot \frac{5}{6} \qquad \text{(term 6-2)}$$

$$+\ \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \qquad \text{(term 6-3)}$$

$$+\ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \qquad \text{(term 6-4)}$$

$$+\ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \qquad \text{(term 6-5)}$$

$$+\ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6}\,. \qquad \text{(term 6-6)}$$

What I will do is to compare the respective terms in the above. Let's not worry about (term 6-6) (the 'left-out' term) for now. Now, I contend

$$\begin{aligned}
\text{(term 5-0)} &= \text{(term 6-0)},\\
\text{(term 5-1)} &= \text{(term 6-1)},\\
\text{(term 5-2)} &< \text{(term 6-2)},\\
\text{(term 5-3)} &< \text{(term 6-3)},\\
\text{(term 5-4)} &< \text{(term 6-4)},\\
\text{(term 5-5)} &< \text{(term 6-5)}.
\end{aligned}$$

In other words, I contend

$$\begin{aligned}
1 &= 1\,,\\
\frac{1}{1}\cdot\frac{5}{5} &= \frac{1}{1}\cdot\frac{6}{6},\\
\frac{1}{1\cdot 2}\cdot\frac{5}{5}\cdot\frac{4}{5} &< \frac{1}{1\cdot 2}\cdot\frac{6}{6}\cdot\frac{5}{6},\\
\frac{1}{1\cdot 2\cdot 3}\cdot\frac{5}{5}\cdot\frac{4}{5}\cdot\frac{3}{5} &< \frac{1}{1\cdot 2\cdot 3}\cdot\frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6},\\
\frac{1}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{5}{5}\cdot\frac{4}{5}\cdot\frac{3}{5}\cdot\frac{2}{5} &< \frac{1}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6}\cdot\frac{3}{6}\,,\\
\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{5}{5}\cdot\frac{4}{5}\cdot\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{1}{5} &< \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6}\cdot\frac{3}{6}\cdot\frac{2}{6}.
\end{aligned}$$

These are actually all true, because

$$\frac{5}{5} = \frac{6}{6}, \qquad \frac{4}{5} < \frac{5}{6}, \qquad \frac{3}{5} < \frac{4}{6}, \qquad \frac{2}{5} < \frac{3}{6}, \qquad \frac{1}{5} < \frac{2}{6},$$

so

$$\frac{5}{5}\cdot\frac{4}{5} < \frac{6}{6}\cdot\frac{5}{6}, \qquad \frac{5}{5}\cdot\frac{4}{5}\cdot\frac{3}{5} < \frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6},$$

$$\frac{5}{5}\cdot\frac{4}{5}\cdot\frac{3}{5}\cdot\frac{2}{5} < \frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6}\cdot\frac{3}{6}, \qquad \text{and}$$

$$\frac{5}{5}\cdot\frac{4}{5}\cdot\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{1}{5} < \frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6}\cdot\frac{3}{6}\cdot\frac{2}{6}.$$

(See ‘ Refresher — Inequalities #5 ’ from “Supplement”.) Now, in the above, (term 6-6) was not involved. But we can see that (term 6-6) is bigger than 0. So, in sum:

$$\begin{array}{rccc}
 & \text{(term 5-0)} & = & \text{(term 6-0)} \\
 & \text{(term 5-1)} & = & \text{(term 6-1)} \\
 & \text{(term 5-2)} & < & \text{(term 6-2)} \\
 & \text{(term 5-3)} & < & \text{(term 6-3)} \\
 & \text{(term 5-4)} & < & \text{(term 6-4)} \\
 & \text{(term 5-5)} & < & \text{(term 6-5)} \\
+) & 0 & < & \text{(term 6-6)} \\
\hline
 & \text{part (5)} & < & \text{part (6)}
\end{array}$$

That’s what I was claiming. So, in short, I was able to compare

(5) $$\left(1+\frac{1}{5}\right)^5, \qquad \text{and}$$

(6) $$\left(1+\frac{1}{6}\right)^6,$$

and successfully concluded that part (6) is bigger than part (5):

$$\left(1+\frac{1}{5}\right)^5 < \left(1+\frac{1}{6}\right)^6.$$

I didn’t use a calculator.

• If you employ the same logic as above, use Binomial Formula, then you will arrive at the conclusion part (7) is bigger than part (6); part (8) is bigger than part (7), and so on. More generally, part $(n+1)$ is bigger than part (n). If you want to be meticulous, and have to convince yourself that that is indeed the case, go as follows:

Expand

(n) $$\left(1+\frac{1}{n}\right)^n, \qquad \text{and}$$

$(n+1)$ $$\left(1+\frac{1}{n+1}\right)^{n+1}$$

each, using Binomial Formula. Call the terms in the resulting expansion of part (n) as $\big(\text{term } n\text{-}k\big)$; $k = 0, 1, 2, 3, \cdots, n$. Call the terms in the resulting expansion of part $(n+1)$ as $\big(\text{term } (n+1)\text{-}k\big)$; $k = 0, 1, 2, 3, \cdots, n+1$. For $k = 0$ and 1 each, $\big(\text{term } n\text{-}k\big)$ and $\big(\text{term } (n+1)\text{-}k\big)$ clearly both equal 1. For $k = 2, 3, 4, \cdots, n$,

- $\big(\text{term } n\text{-}k\big)$ is

$$\frac{1}{1\cdot 2\cdot 3\cdot \ \cdots \ \cdot k}\cdot\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\frac{n-3}{n}\cdot\ \cdots\ \cdot\frac{n-k+1}{n},$$

- $\big(\text{term } (n+1)\text{-}k\big)$ is

$$\frac{1}{1\cdot 2\cdot 3\cdot \ \cdots \ \cdot k}\cdot\frac{n+1}{n+1}\cdot\frac{n}{n+1}\cdot\frac{n-1}{n+1}\cdot\frac{n-2}{n+1}\cdot\ \cdots\ \cdot\frac{n-k+2}{n+1}.$$

We see that $\big(\text{term } (n+1)\text{-}k\big)$ is bigger than $\big(\text{term } n\text{-}k\big)$, because

$$\frac{n}{n} = \frac{n+1}{n+1}, \quad \frac{n-1}{n} < \frac{n}{n+1}, \quad \frac{n-2}{n} < \frac{n-1}{n+1}, \quad \cdots, \quad \frac{n-k+1}{n} < \frac{n-k+2}{n+1}.$$

Finally, for $k = n+1$, $\big(\text{term } (n+1)\text{-}(n+1)\,\big)$ is positive.

So, in sum,

$$\begin{array}{rrcl}
 & (\text{term } n\text{-}0) & = & \big(\text{term } (n+1)\text{-}0\big) \\
 & (\text{term } n\text{-}1) & = & \big(\text{term } (n+1)\text{-}1\big) \\
 & (\text{term } n\text{-}2) & < & \big(\text{term } (n+1)\text{-}2\big) \\
 & (\text{term } n\text{-}3) & < & \big(\text{term } (n+1)\text{-}3\big) \\
 & (\text{term } n\text{-}4) & < & \big(\text{term } (n+1)\text{-}4\big) \\
 & \vdots & & \vdots \\
 & (\text{term } n\text{-}n) & < & \big(\text{term } (n+1)\text{-}n\big) \\
+) & 0 & < & \big(\text{term } (n+1)\text{-}(n+1)\,\big) \\
\hline
 & \text{part } (n) & < & \text{part } \big(n+1\big)
\end{array}$$

This way you have just established

$$\boxed{\left(1+\frac{1}{n}\right)^{n} \quad < \quad \left(1+\frac{1}{n+1}\right)^{n+1}}\ .$$

This is true for every $n = 1, 2, 3, 4, 5, \cdots$.

- So, the calculator did wrong. The actual figures should be

(10^3)	2.7169..,
(10^6)	2.7182804..,
(10^9)	2.718281827..,
(10^{12})	2.718281828..,
(10^{15})	2.718281828..,
(10^{18})	2.718281828..,

even though this fact itself is not apparent from the above argument.

This I know because at least we can trust the calculator's accuracy for part (10^9), part (10^{12}), *etc.* just before the figures start to 'sink'. But I also know it from a more sophisiticated calculating device, a computer software (Maple). It spits out something like

(10^3)	$2.716923932235892457383088121947577 1...$
(10^6)	$\underline{\underline{2.71}}828046931937688381979970845435 63...$
(10^9)	$\underline{\underline{2.71828}}1827099904322376644023860 3328...$
(10^{12})	$\underline{\underline{2.7182818}}2845768609444605919461 41537...$
(10^{15})	$\underline{\underline{2.71828182845}}904387621937324183 12906...$
(10^{18})	$\underline{\underline{2.71828182845904}}52340011465571231398...$
(10^{21})	$\underline{\underline{2.71828182845904523}}535892833043 84329...$
(10^{24})	$\underline{\underline{2.7182818284590452353}}6028611221 17482...$
(10^{27})	$\underline{\underline{2.71828182845904523536028}}746999 35215...$
(10^{30})	$\underline{\underline{2.7182818284590452353602874}}7135 13033...$

What do you notice? It looks like as you move to further and further down in the list, the figures will not grow arbitrarily large, but will get stagnant. I underlined the part in each line that are unchanged from the previous line. But once again, this is computer experiment. My computer might be lying. How are we so sure that the figures cannot grow arbitrarily large? Is there a way to theoretically verify it? My answer is, "yes indeed". What I can theoretically prove is the following:

Claim. The numbers

$$(n) \qquad \left(1+\frac{1}{n}\right)^n,$$

$n = 1, 2, 3, 4, 5, \cdots$, cannot become arbitrarily large. Indeed, the digit before the decimal point in the decimal expression of each of these numbers is 2. In other words, these numbers are all less than 3.

How do you prove that, independently of computer? — To be Continued.