Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – I

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§1. What is mathematics? — The Riemann Hypothesis.

Welcome to Math 105, 'Topics in Mathematics'. On your way to this class, you were thinking oh, you are heading to your math class, and perhaps worrying that maybe someone steals your spot, or you are just running late, and then you spotted me in Jayhawk Boulevard, sigh of relief, but did it occur to you that you would be asked ' what is mathematics ' right off the bat as soon as you walked in to this room?

First of all, 'math' is the short form for the word 'mathematics' and is spelled m-a-t-h, but in United Kingdom it is 'maths', as in m-a-t-h-s. But here, I will just use 'math', without 's'. So, let's talk about some myth of math (no pun intended) right off the bat:

Myth 1. Math is difficult.

— I can read your mind. You want me to say 'no, math is easy, if taught right'. So, here we go, the second myth:

Myth 2. Your professor should be able to explain everything well and math is actually easy. In a perfect world, that's how it goes.

Myth 3. But in the real world, most math professors are not as skilled. That's why we end up hating math. We are good and the system is making us otherwise.

— Here the "system" is the keyword. Let's dissect.

Myth 3′. In math, something is difficult means computationally difficult, not conceptually difficult. Complexity is never what the nature of math calls for. My professor always makes it complex so as to make sure some of us fail. But even the complex ones are solvable by the computer. In other words, math is simple in reality, and complex in disguise.

Myth 3″. In short, math classes are just 'inconvenience' to us. We are learning what's being outdated. Let's let our computer do the job.

Myth 4. So we have created a new standard, which is totally cool. Being bad at math doesn't mean you are dumb. You're normal. Meanwhile, being good at math = moron.

Myth 4′. So math classes are dope. There should be ten math professors in the world, good looking ones, for the YouTube lectures, which nobody is going to watch anyway. Otherwise math professors are just "campus misfits". By the way, the "campus misfits" part was not coming from me, but from Professor Lang.

This is a politically charged topic. As I said earlier, I will eventually address it. Today, let me just say that about one hundred thousand to a quarter million are probably the right numbers for the mathematicians' population in the world, to cover the breadth of research mathematics, just like the number of medical doctors should be at least in that scale. I know you are skeptical. I will address it. But first thing first, let's take another look at

"So, math is deceptively difficult but it is actually easy."

First and foremost, math is a cutting edge science. We certainly have some ideas about blackholes and big-bang theory in astrophysics, and stem cell research in biomedicine. Those are in vogue. What is cutting edge in math? I will give you an example. This is not something your usual math class teaches. But this is widely known, not a secret or anything.

There is a so-called 'The Riemann Hypothesis'. Bernhard Riemann (1826-1866), who seems to continue to earn the split popular vote for the best mathematical genius in the all-time history of math (along with Gauss and Euler, and a few others) has come up with this 'conjecture' (a conundrum). After having encountered a very curious and peculiar phenomenon while investigating what's known today as Riemann's zeta function, Riemann saw that this is very likely true. So he decided to submit this as a conjecture. No one before him had come up with the same. And that was 1859. Little did he know was it would quickly become known as an absolutely impenetrable problem. In fact, It stood for the next 155+ years, had refuted the relentless ultra-intense scrutiny by the top notch mathematical geniuses in the subsequent eras, aided by the most advanced cutting-edge math technology devised specifically by those people to assault it. It still remains as an open problem today, January 21st, 2015.

By the way, computers cannot solve it. I am going to tell you why in a little bit. (That would be my retort to the above raised points at once.) Anybody who solves this problem today would instantly become famous. An organization called Clay Mathematical Institute $\big($CMI$\big)$* offers some prize money on this problem along with a half-dozen other math problems. The amount is one million dollars, actually. While trying not to over-generalize it, money is generally not an incentive for mathematicians. I described to some extent this psyche of mathematicians in the document "Background Information" which I made available to you. By the way, I have the honor to be personally acquainted with one professor, and his disciple who is also a professor, both of whom are the authorities of the Riemann Hypothesis. Their stature is they are among those handful people closest to the ultimate clue to the Riemann Hypothesis. They might actually have a shot at it.

Now, at least a few dozen different possible approaches are known to exist. Of course, nobody knows which one, or any one of them, would work. I myself am not working on the Riemann Hypothesis, as in I am not trying to hit the goldmine and become famous. $\big($Disclaimer: nothing wrong in pursuing fame.$\big)$ It's a gamble because your effort is likely futile given the level of difficulty, and it will ruin your career namely, you would have been successful in working on something else. A common wisdom here is the amount of risks you run and the amount of costs you pay in assaulting a famous problem is extremely high — a high-risk, high-return, deal. Though I do not make a frontal attack, my research expertise partially encompasses this subject, indeed, Riemann's zeta function has such a multi-faceted personality that it infiltrates into my areas of expertise.

Now, quite simply, why is the Riemann Hypothesis so important? Because this has to do with the so-called 'distribution of prime numbers'. Some numbers have the property it cannot be written as a result of multiplying two smaller numbers. Those are called 'prime numbers'. Actually, that is a little imprecise. In fact, any number greater than 1 can be written as a product of two smaller numbers. For example, 5 can be written as

$$5 \;=\; \frac{5}{2}\cdot 2.$$

Yes, this is a legit identity, but this expression of 5 involves a '*non-integer*' $\dfrac{5}{2} = 2.5$. So, I haven't made precise the notion of 'prime numbers' yet. Making it precise necessitates me to specify what exactly kind of numbers should be involved in factoring. There are actually two kinds of numbers. Namely, there are so-called 'integers', and then there are so-called 'non-integers'. I must specify the meaning of those terms. Don't get stressed out. The idea here is that you always have to make sure you leave absolutely no room for ambiguity. That's a part of the nature of mathematics.

* www.claymath.org/millennium-problems

Definition (integers).

(1) Numbers $1,\ 2,\ 3,\ 4,\ \cdots$, are called $\underline{\text{ positive integers }}$.

- If n is a positive integer, then $n+1$ is a positive integer.

(2) Numbers $-1,\ -2,\ -3,\ -4,\ \cdots$, are called $\underline{\text{ negative integers }}$.

- If n is a negative integer, then $n-1$ is a negative integer.
- If n is a positive integer, then $-n$ is a negative integer.
- If n is a negative integer, then $-n$ is a positive integer.

(3) 0 is an integer. 0 is neither a positive integer nor a negative integer.

(4) Positive integers, negative integers, and 0 are called integers. No other numbers are called integers.

So, in short, what you used to call 'whole numbers' are now called 'integers'.

⋆ **Paraphrase.**

An integer is a number such that, in its decimal expression, there are no digits under the decimal point $\big($or all the digits under the decimal point are $0\big)$.

Examples.

- 13 is an integer. Indeed, it is a positive integer.
- -7 is an integer. Indeed, it is a negative integer.
- 1000 is an integer. Indeed, it is a positive integer.
- Once again, 0 is an integer.
- $\dfrac{2}{5} = 0.4$ is not an integer.
- $\dfrac{4}{3} = 1.33333...$ is not an integer.
- $-\dfrac{5}{6} = -0.83333...$ is not an integer.
- $-\dfrac{21}{8} = -2.625$ is not an integer.

$\star$ Don't be misled and think that fractions are not integers. For example,

- $\dfrac{4}{2}$ is an integer. Indeed, this fraction is reduced to 2.
- $\dfrac{-99}{11}$ is an integer. Indeed, this fraction is reduced to -9.

$\star$ **Question.** Why use the term 'integers' instead of 'whole numbers'? This sounds pretentious.

$\star$ **Answer.** That's mainly psychological. The term 'whole number' sounds a little juvenile. 'Integer' sounds more professional and thus is more preferable. No deeper reason. I am going to use this term 'integers' throughout this semester. It simply means 'whole number'.

Now, strictly within the world of positive integers, 5 does not have an expression of the form $5 = a \cdot b$, other than

$$5 = 1 \cdot 5, \qquad \text{and} \qquad 5 = 5 \cdot 1.$$

So 5 is an example of a prime number. Meanwhile, 6 is an example of a non-prime number. Indeed,

$$6 = 2 \cdot 3.$$

More generally:

Definition (prime numbers). A prime number is a positive integer p such that, no matter how you choose a and b, $n = a \cdot b$ cannot be achieved as long as a and b are both positive integers that are strictly smaller than p. So,

$$\begin{aligned}&2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\\ &53,\ 59,\ 61,\ 67,\ \cdots\end{aligned}$$

are prime numbers, whereas

$$\begin{aligned}&1,\ 4,\ 6,\ 8,\ 9,\ 10,\ 12,\ 14,\ 15,\ 16,\ 18,\ 20,\ 21,\ 22,\ 24,\\ &25,\ 26,\ 27,\ 28,\ 30,\ 32,\ 33,\ 34,\ 35,\ 36,\ 38,\ 39,\ 40,\\ &42,\ 44,\ 45,\ 46,\ 48,\ 49,\ 50,\ 51,\ 52,\ \cdots\end{aligned}$$

are <u>not</u> prime numbers.

Pop quiz. The only <u>even</u> prime number is 2. Can you explain why?

So how many prime numbers are there in this world? Around 300 B.C., a mathematician named Euclid in Greece has discovered that there are infinitely many prime numbers. What does that mean? It means as follows. Let p be a prime number. Then no matter how large $\big($or small$\big)$ p is, there is a guarantee there is another prime number strictly bigger than p.

Now, how Euclid convinced himself that is indeed true is an interesting subject in its own right. But let me skip that part. And so far this is very understandable material. But what's so interesting is that, the subsequent research has revealed that there seems no regular patterns in the distribution of prime numbers. Indeed, if you just look at the above list of prime numbers, you might have the impression that the occurence of prime numbers in the integer sequence

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ \cdots$$

is like every fourth one or every sixth one is a prime number. But that is actually going to change before long, namely, the density of prime numbers within the entire integer sequence is going to become much more sparse as you keep climbing up to a larger number territory. The following statement might be counterintuitive to you, but it is true. So listen: There are 1000000 $\big($one million$\big)$ consecutive integers none of which is a prime. By the way, in what follows, 'prime' just means 'prime numbers'. This usage is customary. Worse, in the above statement, you can replace 1000000 $\big($one million$\big)$ with any larger number you like, and you can still find consecutive integers of that length none of which is a prime. There is actually an elementary explanation why that is true. But once again, let me skip that. So, is there any regular patterns the primes exhibit? Mathematicians have tried so hard to discover any sort of patterns for the primes, but failed.

Now, this is where it gets so interesting. A relatively recent research result by Yitang Zhang at University of New Hampshire has revealed that, no matter how large a number you choose, there are two primes above that number and whose gap is less than 70000000 $\big($seventy million$\big)$. That was May, 2013. This is extremely groundbreaking, because before him, no such bound $\big($70000000 is the bound he discovered$\big)$ was proven to exist. Now, it is conjectured that that bound can be sharpened, it is believed to be ultimately trimmed down to 2. This is actually another famous unsolved problem, so-called 'the twin-prime conjecture'. The reason for this naming is it would then be true that there are infinitely many 'twin primes' $\big($= two primes whose gap is precisely 2$\big)$. In view of the content of 'Pop Quiz' above, this 'presumed bound' 2 is optimal, if the twin prime conjecture is indeed true.

Now, I can tell you the reason why computers cannot solve these. These line of problems clearly all deal with infinity. What your computer can tell you is whether up to a certain large number your working hyopthesis is true. But that does not constitute a solution to the problem, unfortunately. Computer's capacity is essentially limited to finite algorithms. Most outstanding problems in math, such as the Riemann Hypothesis, has to do with infinity. Still, computers can be effective tools. Indeed, they serve as a checking device. I too use a computer on a daily basis as a checking device, in my research context. This is the ingrained weakness (or the limitation) of computers. So it's not like fine-tuning their efficiency would fundamentally resolve the matter.

That said, there is one famous example of an outstanding problem that has been solved by computers. Just briefly, it goes something like this. Imagine the map of the contiguous 48 states with the state lines clearly drawn. Then it suffices four different colors to paint the entire map, with one state in one color, and two adjacent states in different colors. And the same conclusion can be drawn for any map, namely, four colors always suffice to paint any map, not just the map of the United States. This was once a famous open problem, and was solved by a team of mathematicians in the 1970s who first wrote up computer programs, and then coaxed their super-computer to solve it.[1] This problem is called 'The Four Color Problem'. Let me emphasize that, since it had been solved, it is not an open problem any more. The reason why the computer way was feasible in this particular case was because the problem was essentially breakable into finite algorithms (steps). Remember, that's the *forte* of computers. Yet please remember that somebody has to coax the computer to do the job, and that part requires expertise, not something a lay person can do. Most outstanding open problems in math are not like this, though. They have to be solved in an *ex machina* way.

And not the Riemann Hypothesis yet. Please bear with me. So, back to the primes, I was telling you that, mathematicians weren't able to identify any sort of meaningful patterns in how the primes are spread across the entire integer sequence. That is, until Riemann. (Actually I am making a long story short here.) Riemann's major discovery, which he thoroughly illustrated in his famous paper submitted as a part of his professorship petition (qualifier) to University of Göttingen, Germany, is that, there is a distinguished function which would later be called Riemann's zeta function $\zeta(s)$, and this function encodes the ultimate clue about the hidden patterns of the distributions of primes.[2] More precisely, our understanding of the distribution of

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[1] Solved by Kenneth Appel and Wolfgang Haken in 1976.

[2] B. Riemann "*Über die Anzahl Primzahlen unter einer gegebenen Größe*" (*On the number of prime bumbers less than a given quantity*), 1859.

primes hinges on the location of s at which that function $\zeta(s)$ takes the zero value. According to Riemann, it is very likely that those locations are lined up in a straight line, called 'the critical line'. Riemann himself did not know how to solve it. What we know is this has been checked for the first 10000000000 (ten billion) zeroes, by computers. But again, that means next to nothing, these are finitely many cases out of infinitely many. Well, actually that was not entirely fair. I shouldn't discredit those who pulled this important computer-aided result.

By the way, the first person who verified that there are indeed infinitely many s at which $\zeta(s)$ takes the zero value on 'the critical line' is called G. H. Hardy (1877-1947), a famous British mathematician and author. Hardy was known to be outspoken, and wrote up a book called "*A Mathematicians' Apology*" wherein he advocated his belief, that the sole value of mathematics is its aesthetic beauty. Hardy was at the center of the British intellectual circle in the early 20th century, and his close friends include the philosopher/mathematician Bertrand Russell, known for the Russell's paradox, and the economist Maynard Keynes, known for the Keynesian economics. We call those s at which $\zeta(s)$ takes the zero value 'the non-trivial zeroes of zeta'. So what Hardy has proved is that there are infinitely many non-trivial zeroes of zeta on 'the critical line'.

⋆ You don't have to know what each of the following means, but let me throw them anyway so you have some idea how Riemann's original paper looks like (mathematicians today understand precisely what these mean):

$$\boxed{\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \left(\mathrm{Re}\ s > 1\right),}$$

$$\boxed{\zeta(s) = \prod_{p:\ \text{prime}} \frac{1}{1 - p^{-s}} \qquad \left(\mathrm{Re}\ s > 1\right),}$$

$$\boxed{\zeta(s) = 2\,\Gamma(1 - s)\,\sin\left(\frac{\pi s}{2}\right) \cdot \left(2\pi\right)^{s-1} \zeta(1 - s).}$$