Math 105 TOPICS IN MATHEMATICS

SOLUTION FOR MIDTERM EXAM – I (Take-home; 03/06)

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⋆ In problem [I] below we work on a model where one can divide any dollar amount by any large number (integer). Also, we never round figures. So, one-third of a dollar is never the same as 33 cents (because 33 cents is one-third of 99 cents).

[I] (Take-home; 20pts) You open a bank account, deposit a dollar in that account.

(1) Your bank offers 100 percent interest annually.

After one year, your balance is $\$\,2$.

(2) Suppose your bank offers a compound interest with 100 percent rate annually.

(2a) After <u>two</u> years, your balance is $\$\,4$.

(2b) After <u>three</u> years, your balance is $\$\,8$.

(3) Suppose the compounding takes place <u>five</u> times annually. So every $\frac{1}{5}$-th of a year $\frac{1}{5}\cdot 100 = 20$ percent of your balance will be accrued as an interest.

(3a) After the $\frac{1}{5}$-th of a year, your balance is $\$\,1.20$.

(3b) After the $\frac{2}{5}$-th of a year, your balance is $\$\,1.44$.

(3c) After one year, your balance is $\$\left(1+\frac{1}{\boxed{5}}\right)^{\boxed{5}}$.

(4) Suppose the compounding takes place 10^{100} times annually. So every $\frac{1}{10^{100}}$-th of a year, $\frac{1}{10^{100}}$ times 100 percent of your balance will be accrued as an interest.

After one year, your balance is $\$ \left(1 + \frac{1}{\boxed{10^{100}}} \right)^{\boxed{10^{100}}}$.

(5) Which one does your answer in (4) fall into?

- ☐ between \$1 and \$2.
- ☐ between \$2 and \$2.50.
- ☐ between \$2.50 and \$3.00.
- ☐ more than \$3.

[Answer]: Between \$ 2.50 and \$ 3.00.

Indeed, the answer in (4) is 2.718281828459045... .

[II] (Take-home; 20pts) (a) Use calculator to pull the decimal expressions of the numbers in each of (a5) through (a10).

(a1) $1 + \frac{1}{1!} = \boxed{2} . \boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}$,

(a2) $1 + \frac{1}{1!} + \frac{1}{2!} = \boxed{2} . \boxed{5}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}$,

(a3) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = \boxed{2} . \boxed{6}\boxed{6}\boxed{6}\boxed{6}\boxed{6}\boxed{6} \ldots$,

(a4) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = \boxed{2} . \boxed{7}\boxed{0}\boxed{8}\boxed{3}\boxed{3}\boxed{3} \ldots$,

(a5) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$

$= \boxed{2} . \boxed{7}\boxed{1}\boxed{6}\boxed{6}\boxed{6}\boxed{6} \ldots$, (a6) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$

$$= \boxed{2}\ .\ \boxed{7}\boxed{1}\boxed{8}\boxed{0}\boxed{5}\boxed{5}\ \ldots\ ,$$

(a7) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!}$

$$= \boxed{2}\ .\ \boxed{7}\boxed{1}\boxed{8}\boxed{2}\boxed{5}\boxed{3}\ \ldots\ ,$$

(a8) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$$= \boxed{2}\ .\ \boxed{7}\boxed{1}\boxed{8}\boxed{2}\boxed{7}\boxed{8}\ \ldots\ ,$$

(a9) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$$+ \frac{1}{9!} = \boxed{2}\ .\ \boxed{7}\boxed{1}\boxed{8}\boxed{2}\boxed{8}\boxed{1}\ \ldots\ ,$$

(a10) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$$+ \frac{1}{9!} + \frac{1}{10!} = \boxed{2}\ .\ \boxed{7}\boxed{1}\boxed{8}\boxed{2}\boxed{8}\boxed{1}\ \ldots\ .$$

(b) Use calculator to find the smallest positive integer n such that

$$\left(1+\frac{1}{n}\right)^n$$

is bigger than the value in (a4) above $\left(= 2.7083333\ldots\right)$.

[Answer]: $n = 136.$ Indeed,

$$\left(1+\frac{1}{135}\right)^{135} \quad = \quad 2.7082819990\ldots\,, \qquad \text{whereas}$$

$$\left(1+\frac{1}{136}\right)^{136} \quad = \quad 2.7083550352\ldots\,.$$

(c) <u>True or false</u> :

"*Let* k *be an arbitrarily chosen positive integer, and fixed. If you choose a large enough* n, *then*

$$\left(1+\frac{1}{n}\right)^{n} \; > \; 1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\frac{1}{5\,!}+\cdots+\frac{1}{k\,!}.\text{''}$$

[<u>Answer</u>]: True.

(d1)

$$e \;=\; \lim_{n\to\infty}\left(1+\boxed{\frac{1}{n}}\right)^{n} .$$

(d2)

$$e \;=\; \lim_{n\to\infty}\left(1+\frac{1}{\boxed{1!}}+\frac{1}{\boxed{2!}}+\frac{1}{\boxed{3!}}+\frac{1}{\boxed{4!}}+\cdots+\frac{1}{\boxed{n!}}\right).$$

(d3) The decimal expression of e up to the first six place under the decimal point

$$\boxed{2}\;.\;\boxed{7}\boxed{1}\boxed{8}\boxed{2}\boxed{8}\boxed{1}\;\ldots\;.$$

[III] (Take-home; 20pts) Prove that $\sqrt{3}$ is an irrational number.

Proof. Proof by contradiction. Suppose $\sqrt{3}$ is written as

$$\sqrt{3} \;=\; \frac{k}{m}$$

using some integers k and m $(\text{where } m \neq 0)$.

First, if both k and m are divisible by $\boxed{3}$, then we may simultaneously divide both the numerator and the denominator by $\boxed{3}$ (and the value of the fraction stays the same). After that procedure, suppose both the numerator and the denominator still remain to be divisible by $\boxed{3}$, then we repeat the same procedure as many times as necessary until at least one of the numerator and the denominator is not divisible by $\boxed{3}$. Thus we may assume, without loss of generality, that at least one of k and m is $\underline{\text{ not divisible by } 3 \ }$.

Under this assumption, square the both sides of the identity $\sqrt{3} = \dfrac{k}{m}$, thus

$$3 = \frac{\boxed{k^2}}{\boxed{m^2}}.$$

This is the same as

$$\boxed{3m^2} = k^2.$$

The left-hand side of this last identity is clearly divisible by $\boxed{3}$, so this last identity forces its right-hand side to be $\underline{\text{ divisible by } 3 \ }$.

That in turn implies k is $\underline{\text{ divisible by } 3 \ }$, because if k is $\underline{\text{ not divisible by } 3 \ }$, then k^2 is $\underline{\text{ not divisible by } 3 \ }$.

But then k being divisible by $\boxed{3}$ implies k^2 is $\underline{\text{ divisible by } 9 \ }$.

So by virtue of the above last identity $3\,m^2$ is divisible by $\boxed{9}$, or the same to say, m^2 is divisible by $\boxed{3}$. This implies that m is $\underline{\text{ divisible by } 3 \ }$.

In short, both k and m are $\underline{\text{ divisible by } 3 \ }$.

This contradicts our assumption. The proof is complete. $\square$