Math 105 TOPICS IN MATHEMATICS

SOLUTION FOR MIDTERM EXAM – I (In-class; 03/11)

March 13 (Fri), 2015

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Line #: 52920.

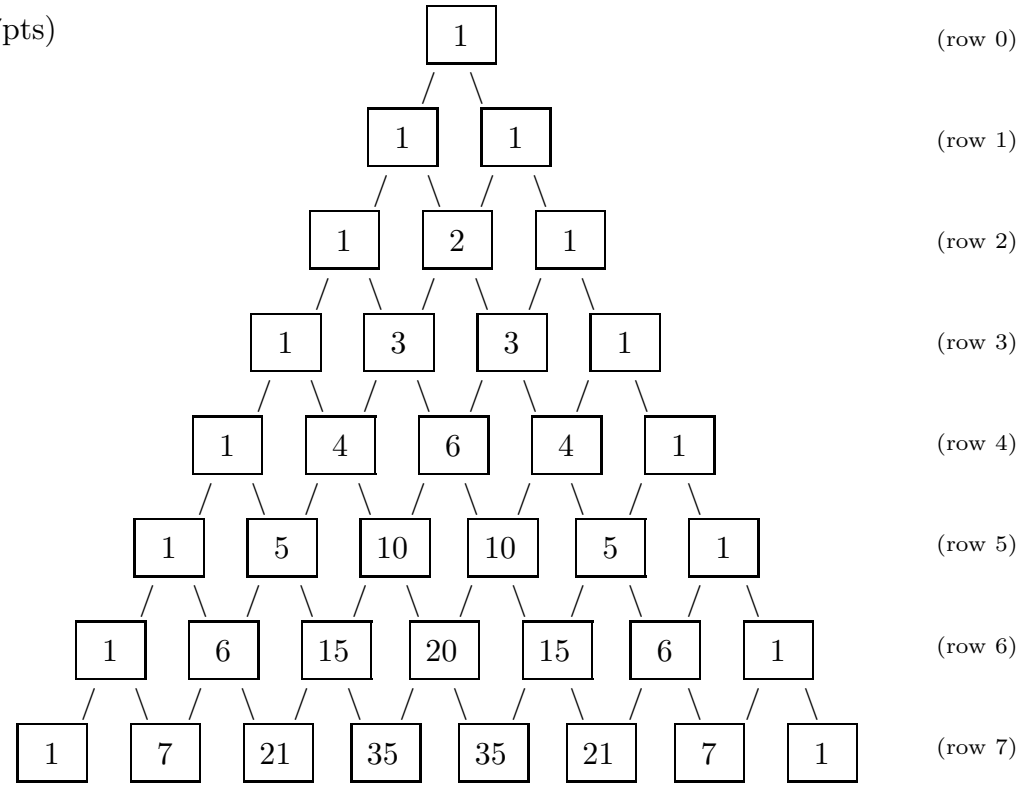
[I] (18pts) (1) Are there infinitely many, or finitely many, prime numbers?

[**Answer**]**:** Infinitely many.

(2) List all the prime numbers between 2 and 50.

[Answer]: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

[II] (7pts)



[III] (3pts) Complete the identities:

(1) $\binom{9}{3} = \dfrac{9 \cdot 8 \cdot 7}{\boxed{3}\,!}$. (2) $\binom{15}{4} = \dfrac{\boxed{15} \cdot \boxed{14} \cdot \boxed{13} \cdot \boxed{12}}{4\,!}$.

[IV] (4pts) You started a lemonade business. Words spread fast and the sales dramatically increased (Table 1 below):

day	$
1	1
2	2
3	6
4	24
5	120
6	720
⋮	⋮

On Day n, your sales will be n times the previous day sales. Suppose the same patterns hold until Day 90. Then the sales on Day 90 is how much? The answer involves '!' (the factorial symbol). You don't have to simplify (calculate) the answer.

[Answer]: 90! .

[V] (7pts) Circle all 2-to-the-powers among the numbers listed below.

4, 6, 8, 10, 12, 16, 24, 32, 36, 48,

54, 64, 72, 84, 96, 108, 128, 144, 216, 256.

[Answer]: 4, 8, 16, 32, 64, 128, 256.

[VI] (8pts) Simplify:

(1) $\sqrt[4]{16} = 2.$ (2) $\sqrt{7} \cdot \sqrt{5} = \sqrt{35}.$

(3) $\sqrt{\sqrt{3}} = \sqrt[4]{3} \left(= 3^{\frac{1}{4}} \right).$ (4) $100^0 = 1.$

[VII] (3pts)

(1) $\left(x+y\right)^4 = x^4 + \boxed{4}\, x^3 y + \boxed{6}\, x^2 y^2 + \boxed{4}\, x y^3 + y^4 .$

(2) If $a + b + c = 0$ then $a^3 + b^3 + c^3 = \boxed{3}\, a\, b\, c.$

[VIII] (4pts) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = \frac{127}{128}.$

$\left(\text{Also,} \quad 1 - \frac{1}{128} \quad \text{is acceptable as an answer.}\right)$

[IX] (6pts) (1) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55.$

(2)
$$\begin{aligned}
&1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
&+ 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 \\
&+ 21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 \\
&+ 31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39 + 40 \\
&+ 41 + 42 + 43 + 44 + 45 + 46 + 47 + 48 + 49 + 50 \\
&+ 51 + 52 + 53 + 54 + 55 + 56 + 57 + 58 + 59 + 60 \\
&+ 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70 \\
&+ 71 + 72 + 73 + 74 + 75 + 76 + 77 + 78 + 79 + 80 \\
&+ 81 + 82 + 83 + 84 + 85 + 86 + 87 + 88 + 89 + 90 \\
&+ 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 + 100
\end{aligned}$$

$= 101 \cdot 50 = 5050.$

[X] (Extra 3pts) Babylonians knew that

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = \frac{30547}{21600}$$

gives a good approximation of $\sqrt{2}$. Indeed,

(1) $$\frac{30547}{21600} = 1.41421296296296... ,$$

(2) $$\sqrt{2} = 1.41421356237309... .$$

The difference between line (1) and line (2) is less than $\frac{1}{60^3}$.

Now, fill in a concrete integer between 0 and 59 into the box below, so that the difference between $\sqrt{2}$ and line (3) below does not exceed $\frac{1}{60^4}$:

(3) $$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} + \frac{\square}{60^4}.$$

$\underline{\text{Answer}}$: $$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} + \frac{\boxed{7}}{60^4}.$$

Reason: Using calculator, you get

$$\left(\sqrt{2} - \left(1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \right) \right) \cdot 60^4 = 7.7683553...$$

Round down this number at the decimal point, so the answer is 7.

[XI] (Extra 7pts) In the formula

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n\cdot\Big(n+1\Big) \;=\; \frac{1}{3}\,n\Big(n+1\Big)\Big(n+2\Big)$$

$\Big($essentially covered in class$\Big)$, substitute $n = 9$:

(1) $\quad 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6 + 6\cdot 7 + 7\cdot 8 + 8\cdot 9 + 9\cdot 10 \;=\; 330.$

Meanwhile,

(2) $\quad 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \;=\; 45.$

Subtract line (2) from line (1), and thereby evaluate

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2.$$

$\Big[$ Answer $\Big]$: $\quad 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 \;=\; 285.$

Work.

$$\begin{array}{cccccccccccccccccc}
 & \underbrace{1\cdot 2} & + & \underbrace{2\cdot 3} & + & \underbrace{3\cdot 4} & + & \underbrace{4\cdot 5} & + & \underbrace{5\cdot 6} & + & \underbrace{6\cdot 7} & + & \underbrace{7\cdot 8} & + & \underbrace{8\cdot 9} & + & \underbrace{9\cdot 10} & \\
 & \| & & \| & & \| & & \| & & \| & & \| & & \| & & \| & & \| & \\
 & \boxed{2} & + & \boxed{6} & + & \boxed{12} & + & \boxed{20} & + & \boxed{30} & + & \boxed{42} & + & \boxed{56} & + & \boxed{72} & + & \boxed{90} & = & \boxed{330} \\
 & \boxed{1} & + & \boxed{2} & + & \boxed{3} & + & \boxed{4} & + & \boxed{5} & + & \boxed{6} & + & \boxed{7} & + & \boxed{8} & + & \boxed{9} & = & \boxed{45} \\
-) & \multicolumn{17}{c}{\hrulefill} \\
 & \boxed{1} & + & \boxed{4} & + & \boxed{9} & + & \boxed{16} & + & \boxed{25} & + & \boxed{36} & + & \boxed{49} & + & \boxed{64} & + & \boxed{81} & = & \boxed{285} \\
 & \| & & \| & & \| & & \| & & \| & & \| & & \| & & \| & & \| & \\
 & 1^2 & & 2^2 & & 3^2 & & 4^2 & & 5^2 & & 6^2 & & 7^2 & & 8^2 & & 9^2 &
\end{array}$$